EFN 307G + 315G

(1)

(1) Ljóseindagas $u = \left(\frac{4\sigma}{c}\right)T^4$, $pV = \frac{U}{3}$

a) pV- og TS-myndir $\to U = \left(\frac{4\sigma}{c}\right)VT^4$ $p = \frac{u}{3} = \frac{1}{3}\left(\frac{4\sigma}{c}\right)T^4$

Fyrsta lögmálið $dU = TdS - pdV$

$$\to dS = \frac{dU + pdV}{T}$$ ← viljum finna $S(T,V)$

$$dU = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT$$

$$= \left(\frac{4\sigma}{c}\right)T^4 dV + \frac{16\sigma}{c}VT^3 dT$$

$$\to dS = \left(\frac{4\sigma}{c}\right)T^3 dV + \frac{16\sigma}{c}VT^2 dT + \frac{1}{3}\left(\frac{4\sigma}{c}\right)T^3 dV$$

(2)

$$\to dS = \frac{16\sigma}{c}VT^2 dT + \frac{4}{3}\left(\frac{4\sigma}{c}\right)T^3 dV$$

$$= \frac{16\sigma}{c}VT^2 dT + \frac{16\sigma}{3c}T^3 dV$$

heildum $\to S(T,V) = \frac{16\sigma}{3}VT^3 + S_0$

$\to$ $\boxed{\text{fast } S \text{ jafngildir } VT^3 = \text{fasti}}$

Höfum líka $p = \frac{4\sigma}{3c}T^4 \to T = \left(\frac{3cp}{4\sigma}\right)^{1/4}$

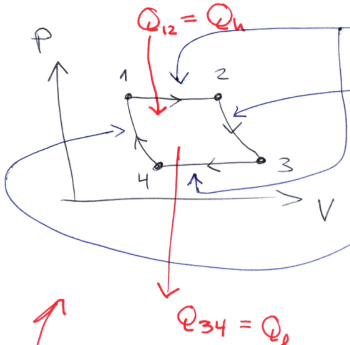
eða $V\cdot T^3 = V\left(\frac{3cp}{4\sigma}\right)^{3/4} = $ fasti

eða $\boxed{p^{3/4}V = \text{fasti}}$ eða $\boxed{pV^{4/3} = \text{fasti}}$

þegar S er fast

(3)

Vegna $p = \frac{4\sigma}{3c}T^4$ þá eru jafnhitaferlar líka jafnþrýstiferlar



jafngengir óvermnir ferlar

$\to dS = 0$, fast S

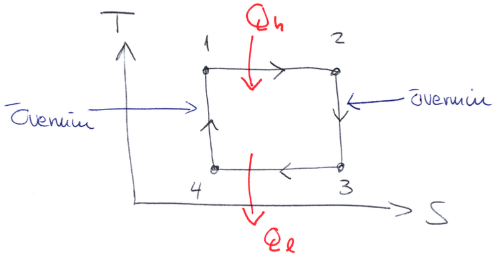
$pV^{4/3} = $ fasti

$T_h = T_1 = T_2$

$T_\ell = T_3 = T_4$

$p_2V_2^{4/3} = p_3V_3^{4/3}$

$p_1V_1^{4/3} = p_4V_4^{4/3}$



(4)

b) Jafngengnir óvermnir ferlar $V\cdot T^3 = $ fasti því $\Delta S = \frac{16\sigma}{3}VT^3$

þekkjum T_ℓ og T_h

$$\to V_3 = V_2\left(\frac{T_h}{T_\ell}\right)^3 \quad \text{og} \quad V_4 = V_1\left(\frac{T_h}{T_\ell}\right)^3$$

c) Vinnan út   p er fasti   því $p = \frac{4\sigma}{3c}T^4$

$$W_{12} = W_h = P_1\cdot(V_2 - V_1) = \frac{4\sigma}{3c}T_h^4\cdot(V_2 - V_1)$$

$$Q_h = T_h\cdot(S_2 - S_1) = \frac{16\sigma}{3c}T_h^4(V_2 - V_1)$$

varmi inn

5

Vinna inn

$$W_{34} = W_e = P_3 \cdot (V_3 - V_4) = \frac{4\sigma}{3c} T_e^4 (V_3 - V_4)$$

varmi út

$$Q_{34} = Q_e = \frac{16\sigma}{3c} T_e^4 (V_3 - V_4) = \frac{16\sigma}{3c} T_e T_h^3 (V_2 - V_1)$$

því $V_3 = V_2 \left(\frac{T_h}{T_e}\right)^3$

$V_4 = V_1 \left(\frac{T_h}{T_e}\right)^3$

Vinna út

$$W_{23} = U_2 - U_3 = \frac{4\sigma}{c}\left(V_2 T_h^4 - V_3 T_e^4\right)$$

$$= \frac{4\sigma}{c}\left(V_2 T_h^4 - V_2 \left(\frac{T_h}{T_e}\right)^3 T_e^4\right) = \frac{4\sigma}{c} V_2 T_h^3 (T_h - T_e)$$

Vinna inn

$$W_{41} = U_1 - U_4 = \frac{4\sigma}{c}\left(V_1 T_h^4 - V_4 T_e^4\right) = \frac{4\sigma}{c} V_1 T_h^3 (T_h - T_e)$$

6

d) nýtni

$$\eta = \frac{W}{Q_h} = \frac{W_{12} + W_{23} - W_{34} - W_{41}}{Q_{12}}$$

Q_h

$$= \frac{\frac{4\sigma}{3c} T_h^4 (V_2 - V_1) + \frac{4\sigma}{c} V_2 T_h^3 (T_h - T_e) - \frac{4\sigma}{3c} T_e^4 (V_3 - V_4) - \frac{4\sigma}{c} V_1 T_h^3 (T_h - T_e)}{\frac{16\sigma}{3c} T_h^4 (V_2 - V_1)}$$

$$= \frac{\frac{4\sigma}{3c} T_h^4 (V_2 - V_1) + \frac{4\sigma}{c} V_2 T_h^3 (T_h - T_e) - \frac{4\sigma}{3c} T_e T_h^3 (V_2 - V_1) - \frac{4\sigma}{c} V_1 T_h^3 (T_h - T_e)}{\frac{16\sigma}{3c} T_h^4 (V_2 - V_1)}$$

$$= \frac{1}{4} - \frac{T_e}{4T_h} + \frac{T_h - T_e}{\frac{4}{3} T_h} = 1 - \frac{T_e}{T_h}$$

eins og við mátti búast fyrir Carnot

7

EFN 307G + 315G

2

$T_1 = 10\,K, \quad T_2 = 20\,K, \quad C_p(T) = aT^3$

a) Hve mikla orku í hitunina?

$$Q_{12} = M \int_{T_1}^{T_2} dT' C_p(T') = \frac{Ma}{4}\left(T_2^4 - T_1^4\right)$$

$$M = 10g = 0{,}01\,kg, \quad a = \frac{30.5}{(348)^3} \frac{kJ}{kg \cdot K^4}$$

$$\rightarrow Q_{12} = \frac{0{,}01}{4} \frac{30.5}{(348)^3}\left(20^4 - 10^4\right) = 2.7 \cdot 10^{-4}\,kJ$$

$$= 0.27\,J$$

b) Minnsta orka til að kæla koparinn aftur

Fyrsta lögmálið $\Delta U = \Delta Q + \Delta W$

8

því búumst við við að

$$\Delta W \geq \Delta U - \Delta Q$$

þar sem "=" myndi gilda fyrir jafngengt ferli

vitum $\Delta U = -0{,}27\,J$

umhverfishitinn

$$\Delta W \geq U(1) - U(2) - T_R\left\{S(T_1) - S(T_2)\right\}$$

$$S(T_1) - S(T_2) = M \int_{T_2}^{T_1} dT' \frac{C_p(T')}{T} = \frac{Ma}{3}\left(T_1^3 - T_2^3\right)$$

$$\rightarrow -\Delta Q = -T_R\left\{S(T_1) - S(T_2)\right\} = -\frac{Ma}{3} T_R \left(T_1^3 - T_2^3\right)$$

$$-\Delta Q = -\frac{0.01}{3}\,\frac{305}{(348)^3}\,293\cdot\left(10^3-20^3\right)\,kJ \simeq 4.95\,J$$

$$\to W \geq \Delta U - T_R\Delta S = -0.27\,J + 4.95\,J = 4.68\,J$$

EFN307G

(3) 3D skammtagas, óvíxlverkandi fermi eða bóseindir

a) bós-gas $T<T_c$, $z\sim 1$

þurfum varmafræðilega mættið Φ_G því

$$p = -\left(\frac{\partial \Phi_G}{\partial V}\right)_{T,\mu}$$

sleppum spuna $\to$

$$\Phi_G = k_BT\frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty dE\sqrt{E}\,\ln\left\{1-e^{-\beta(E-\mu)}\right\} + \Phi_0$$

Φ_0 er vegna stórsærrar setni grunnástandsins $E=0$ því $T<T_c$, en $E=0$ jafngildir líka $\hbar k=0$, hverfandi skriðþunga $\to$ Φ_0 gefur engan þrýsting!

fyrir $T<T_c$ notum við $z\sim 1$ eða $\mu\simeq 0$

$$\to p = -k_BT\frac{1}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty dE\sqrt{E}\,\ln\left\{1-e^{-\beta E}\right\}$$

breytuskipti $x=\beta E$

$$p = -\frac{(k_BT)^{5/2}}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty dx\sqrt{x}\,\ln\left\{1-e^{-x}\right\}$$

$$= -\frac{(k_BT)^{5/2}}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left\{-\frac{2}{3}\Gamma\left(\frac{5}{2}\right)\zeta\left(\frac{5}{2}\right)\right\} = \frac{(k_BT)^{5/2}}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}0.67\sqrt{\pi}$$

p er óháð V vegna þess að fyrir neðan T_c fara umfram eindir einfaldlega í grunnástandið þegar rúmmálið er minnkað

b) vegna þess að $E=0$ er mest einsetið og leiðréttingin vegna þess $\frac{1}{N}$ er hverfandi fyrir stórt N, mikinn fjölda einda.

EFN307G-315G

(4) $E_n = \epsilon_0 + n\epsilon$, $n = 0,1,2,\dots,n_0-1$

Kórsumma

$$Z = \sum_{n=0}^{n_0-1}\exp\{-\beta(\epsilon_0+n\epsilon)\} = e^{-\beta\epsilon_0}\sum_{n=0}^{n_0-1}e^{-\beta n\epsilon}$$

$$Z = e^{-\beta\epsilon_0}\sum_{n=0}^{n_0-1}\left(e^{-\beta\epsilon}\right)^n = e^{-\beta\epsilon_0}\left\{\frac{1-e^{-\beta\epsilon n_0}}{1-e^{-\beta\epsilon}}\right\}$$

Sem stefnir á kórsummuna fyrir hreintóna sveifilinn þegar $n_0\to\infty$.

a) $$\langle E\rangle = -\frac{\partial \ln Z}{\partial\beta} = \epsilon_0 - \frac{\epsilon n_0 e^{-\beta\epsilon n_0}}{1-e^{-\beta\epsilon n_0}} + \frac{\epsilon e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}}$$

$$= \epsilon_0 + \frac{\epsilon}{e^{\beta\epsilon}-1} - \frac{\epsilon n_0}{e^{\beta\epsilon n_0}-1}$$

Athugum að eins $E = \epsilon_0 + n\epsilon$

$\to \langle E\rangle = \epsilon_0 + \langle n\rangle\epsilon$ og því

$$\langle n\rangle = \frac{1}{e^{\beta\epsilon}-1} - \frac{n_0}{e^{\beta\epsilon n_0}-1}$$

sem hefur rétt markgildi þegar $n_0\to\infty$

b) Reikna S og C_V fyrir sveifilinn

$$F = U - TS = E - TS \to S = \frac{E-F}{T}$$

$$F = -k_B T \cdot \ln Z$$

$$\to S = \frac{E}{T} + k_B \ln Z$$

$$= k_B \left\{ \ln\left(1-e^{-\beta\epsilon n_0}\right) - \ln\left(1-e^{-\beta\epsilon}\right) \right\} - \frac{\epsilon}{T}\left\{ \frac{n_0}{e^{\beta\epsilon n_0}-1} - \frac{1}{e^{\beta\epsilon}-1} \right\}$$

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = -\beta\left(\frac{\partial S}{\partial \beta}\right)_V = k_B \beta^2 \epsilon^2 \left\{ \frac{e^{\beta\epsilon}}{(e^{\beta\epsilon}-1)^2} - \frac{n_0^2 e^{\beta\epsilon n_0}}{(e^{\beta\epsilon n_0}-1)^2} \right\}$$

Ef $k_B T \ll n_0 \epsilon$ þá koma aðeins lægstu stigin við sögu og afstýfði sveifillinn hagar sér eins og hreintóna sveifill $C_V \to 0$

Aðal munurinn kemur fram við hátt hitastig

$$k_B T \gg n_0 \epsilon \to \beta\epsilon n_0 \ll 1$$

$$(\beta\epsilon)^2 \frac{e^{\beta\epsilon}}{(e^{\beta\epsilon}-1)^2} \xrightarrow[\beta\epsilon \to 0]{} 1$$

$$(\beta\epsilon n_0)^2 \frac{e^{\beta\epsilon n_0}}{(e^{\beta\epsilon n_0}-1)^2} \xrightarrow[\beta\epsilon n_0 \to 0]{} 1$$

$$\to C_V \xrightarrow[\beta\epsilon n_0 \to 0]{} 0$$

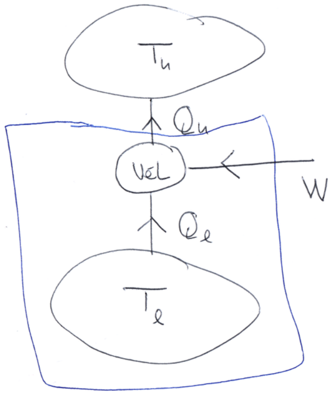
En fyrir hreintóna sveifilinn fékkst $C_V \to k_B$

Hreintóna sveifillinn hefur alltaf hærri stig til að taka við orkunni með hækkandi T, en ekki sá afstýfði!

EFN315G

(3) Ísskápur

$$\eta = \frac{Q_l}{W}$$



ísskápur

bestu nýtni næði ísskápur með Carnot vél

$$\eta_{Carnot} = \frac{T_l}{T_h - T_l}$$

$$\eta \leq \eta_{Carnot}$$

$$\frac{Q_L}{W} \leq \frac{T_l}{T_h - T_l}$$

jafn góður og Carnot vélar

$$Q_l \geq W \to \frac{T_l}{T_h - T_l} \geq 1 \to T_l \geq T_h - T_l$$

eða $2T_l \geq T_h \to T_l \geq \frac{T_h}{2} = 146{,}5\,\mathrm{K} = -126.5^\circ\mathrm{C}$