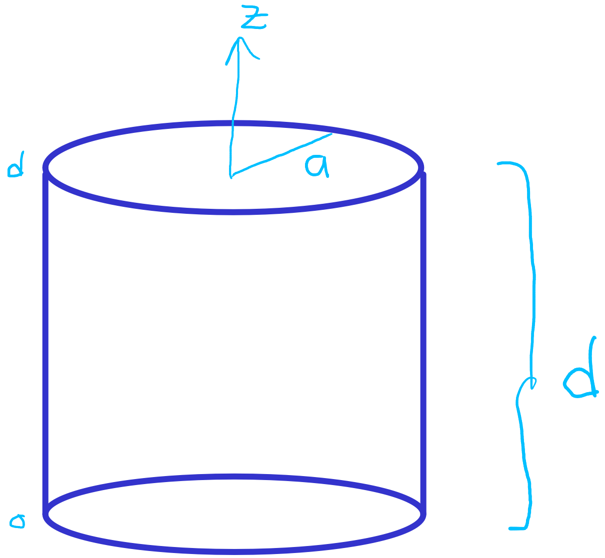
Problem set 10



Cylindrical EM-cavity with TE mode

$m = 0, 1, 2, \dots$
$n = 1, 2, \dots$
$p = 1, 2, \dots$

$$H_z(r, \phi, z) = B_{mnp}\, J_m\left(\chi'_{mn} \frac{r}{a}\right) \cos(m\phi) \sin\left(p\pi \frac{z}{d}\right)$$

$$\bar{E}_T = \frac{i}{h^2} \omega \mu\, \hat{a}_z \times \bar{\nabla}_T H_z$$

χ_{mn} is the n'-th root of the derivative of the m-th Bessel Function

For TE_{011}

$$\boxed{\begin{aligned} H_z &= B_{011}\, J_0\left(\chi'_{01} \frac{r}{a}\right) \sin\left(\frac{\pi z}{d}\right) \\ E_\phi &= B_{011}\, J_0'\left(\chi'_{01} \frac{r}{a}\right) \sin\left(\frac{\pi z}{d}\right) \frac{i}{h} \omega \mu \end{aligned}}$$

$$E_r = \frac{i\omega\mu}{h^2} B_{011}\, J_0\left(\chi'_{01} \frac{r}{a}\right) \sin(0) \cdots = 0$$

$$\bar{\nabla}_T = \hat{a}_r \partial_r + \hat{a}_\phi \frac{1}{r} \partial_\phi$$

When the cavity is closed (bottom, top) we get

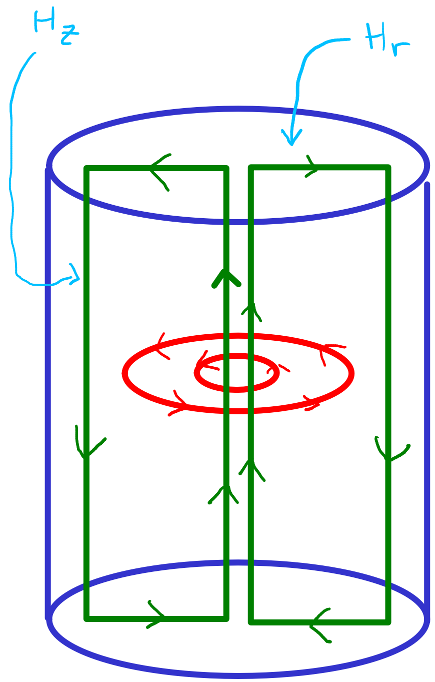
$$H_r \sim J_0\left(\chi'_{01} \frac{r}{a}\right) \cos\left(\frac{\pi z}{d}\right)$$

$$J_0'(x) = -J_1(x) \quad , \quad J_1(0) = 0 \quad , \quad J_1(x) \xrightarrow[x \to 0]{} \frac{x}{2} \quad \text{Linear}$$

$$\longrightarrow \quad E_\phi \sim \underbrace{J_1\left(\chi_{11} \frac{r}{a}\right)} \underbrace{\sin\left(\frac{\pi z}{d}\right)} \qquad J_0(0) = 1$$

Makes the electric field vanish at the curved surface and the top and bottom

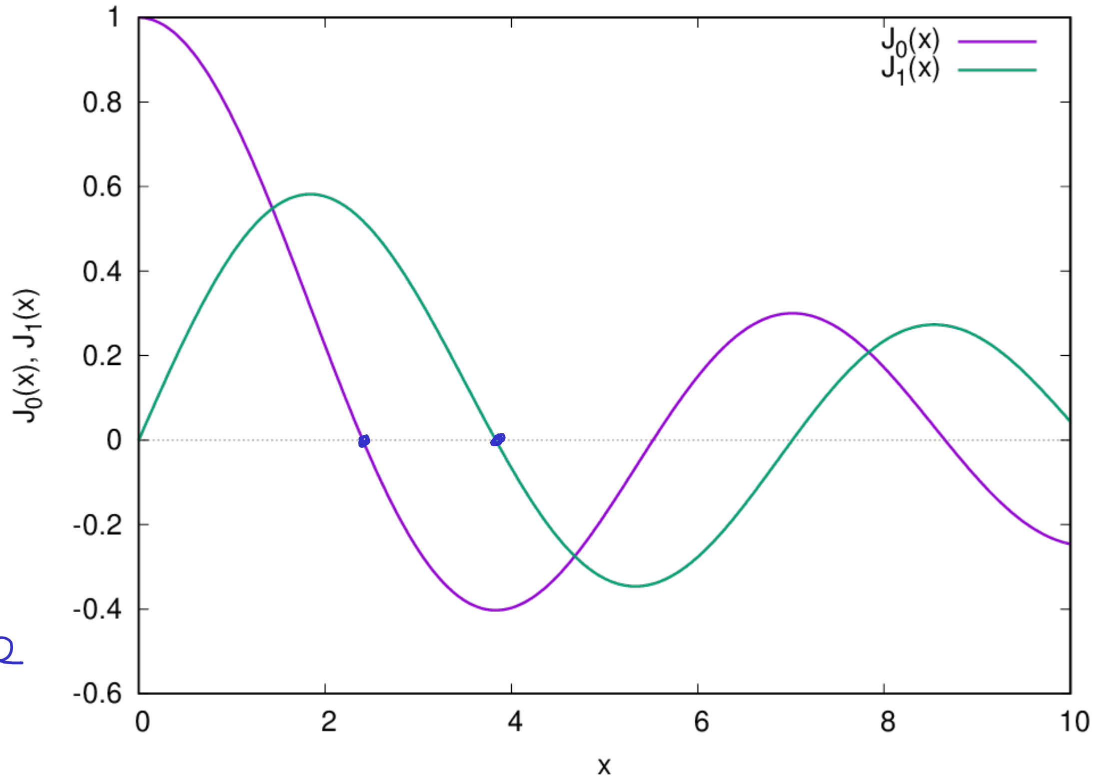
The magnetic field is everywhere parallel to the surfaces, inducing current densities on them



Note, J_0 changes sign before J_1 hits a zero

$\downarrow$

H_z turns around close to the curved surface



In the center $\quad r \ll a, \; z \sim \frac{d}{2}$

$$\bar{\nabla} \times \bar{A} = \hat{a}_z \frac{1}{r} \left[\frac{\partial}{\partial r} r A_\phi - \underbrace{\frac{\partial A_r}{\partial \phi}}_{=0} \right] = \hat{a}_z \frac{1}{r} \frac{\partial}{\partial r} r A_\phi$$

and $\quad \bar{E} = -\frac{1}{c} \partial_t \bar{A}$

Correspondence

use

$$\frac{d}{dx}\left(x^n J_n(x) \right) = x^n J_{n-1}(x)$$

$$\rightarrow \frac{1}{x} \frac{d}{dx}\left(x J_1(x) \right) = J_0(x)$$

$$\rightarrow A_\phi \sim J_1(x) \quad \text{and} \quad E_\phi \sim J_1(r) \xrightarrow[r \to 0]{} \frac{r}{2}$$

So, a vector potential of the form

$$\bar{A} = \hat{a}_\phi A_0 \left(\frac{r}{a} \right)$$

can be used to generate $\bar{E}$ and $\bar{H}$ in the center of the cavity

We see $\bar{B} = \bar{\nabla} \times \bar{A} \sim \hat{a}_z \cdot \text{Constant}$

$$\bar{E} \sim \hat{a}_\phi \left(\frac{r}{a}\right)$$

Close to the center of the cavity the circular electric field and the constant magnetic field couple well with electrons in a 2D sheet perpendicular to the axis of the cylinder and promote circular currents in the system. An external magnetic field along the z-axis, constant in time, would make the cavity mode ideal to promote magnetic transitions, no electric dipole oscillations are promoted by the cavity field.

Remember, the cavity fields oscillate harmonically in time