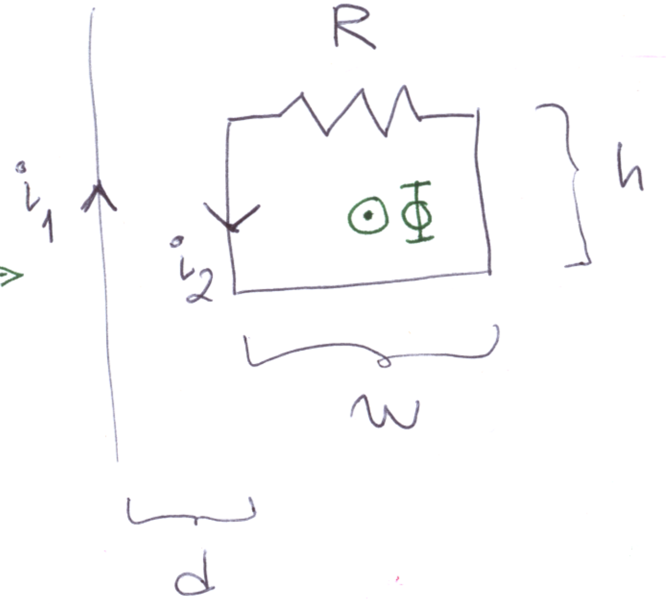
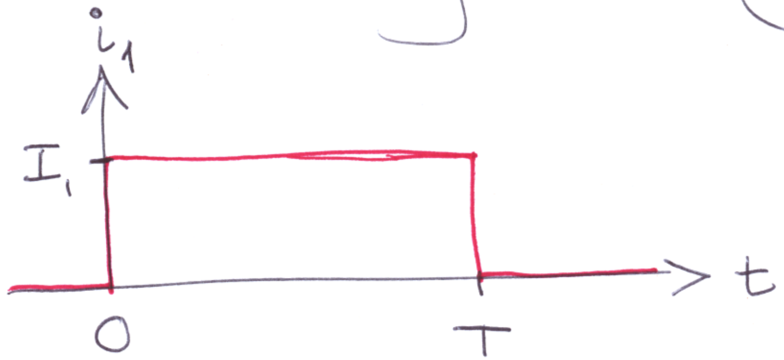
Problem 1



We define the currents as shown in the diagram. Square pulse is sent along the straight wire



a) find the induced current i_2 in the loop with L

b) find the energy dissipated in R if $T >> L/R$

i_1 and i_2 create flux in opposite directions

We will use Faraday's law of induction, but ~~we~~ need to define the flux through the Loop:

$$\Phi_2 = \Phi_{12} + \Phi_{22}$$

$$= -L_{12}\, i_1 + L\, i_2$$

I define positive flux out of the page

$$\oint_{C_2} \bar{E}\cdot d\bar{l} = -\frac{d\Phi_2}{dt} = +L_{12}\frac{di_1}{dt} - L\frac{di_2}{dt}$$

$$\hookrightarrow \; = R i_2$$

$$\rightarrow \quad R i_2 = +L_{12}\frac{di_1}{dt} - L\frac{di_2}{dt}$$

$$\rightarrow \quad L_{12}\frac{di_1}{dt} - L\frac{di_2}{dt} - R i_2 = 0 \qquad (*)$$

L is given, but we need to find L_{12}

$$L_{12} = \frac{\Phi_{12}}{i_1} = \frac{h}{i_1}\int_d^{d+w} dr\, B_{12} = \frac{h}{i_1}\int_d^{d+w} \frac{\mu_0 i_1}{2\pi r}\, dr = \frac{\mu_0 h}{2\pi}\ln\left(1+\frac{w}{d}\right)$$

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a) We need to evaluate

$$\frac{di_1}{dt} = I_1 \frac{d}{dt}\left[\theta(t)\,\theta(T-t)\right], \qquad \theta(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t>0 \end{cases}$$

Heaviside step function

We have to use

$$\frac{d\theta(t)}{dt} = \delta(t) \leftarrow \text{Dirac-delta function}$$

$$\rightarrow \frac{di_1}{dt} = I_1 \left\{\delta(t)\,\theta(T-t) - \theta(t)\,\delta(t-T)\right\}$$

$$= I_1 \left\{\delta(t) - \delta(t-T)\right\}$$

So Faraday's law (*) gives us the differential equation

$$\boxed{\frac{di_2}{dt} + \frac{R}{L} i_2 = \frac{L_{12}}{L} I_1 \left\{\delta(t) - \delta(t-T)\right\}}$$

Inhomogeneous first order equation of the form

$$y' + p(t)y = q(t)$$

has the solution

$$y(t) = y(t_0)\,e^{-P(t)} + e^{-P(t)}\int_{t_0}^{t} e^{P(s)}\,q(s)\,ds$$

Where

$$\underline{P}(t) = \int_{t_0}^{t} ds\, p(s)$$

We have here

$$P(t) = \int_{0}^{t} ds\,\frac{R}{L} = \frac{Rt}{L}$$

and our solution of the inhomogeneous differential equation is

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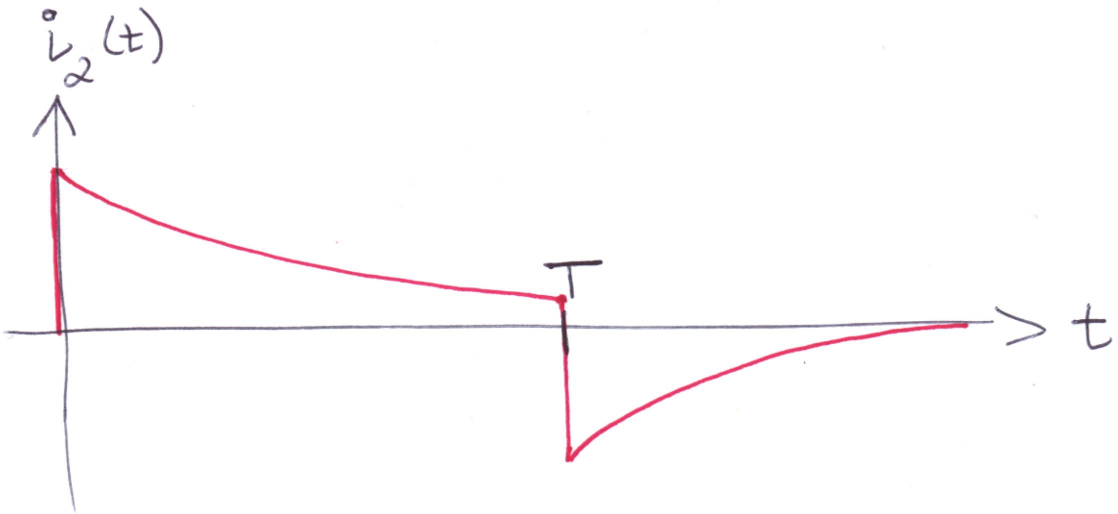
$$i_2(t) = i_2(0)\, e^{-\frac{Rt}{L}} + \frac{L_{12} I_1}{L} \left\{ e^{-\frac{Rt}{L}} - \theta(t-T)\, e^{-\frac{R}{L}(t-T)} \right]$$

use $\quad i_2(0) = 0$

$$\rightarrow \quad i_2(t) = \frac{L_{12} I_1}{L} \left\{ e^{-\frac{Rt}{L}} - \theta(t-T)\, e^{-\frac{R}{L}(t-T)} \right]$$



If $T \gg \frac{L}{R} \rightarrow e^{-\frac{RT}{L}} \ll 1$

and the solution after $t > T$

is thus

asymptotic form

$$i_2(t) \simeq -\frac{L_{12} I_1}{L} e^{-\frac{R}{L}(t-T)}$$

b) The energy dissipated in R of $T \gg \frac{L}{R}$

$$E_{diss} = \int_0^\infty dt\, i_2^2(t) \cdot R \simeq \left(\frac{L_{12} I_1}{L}\right)^2 R \int_0^\infty e^{-\frac{2Rt}{L}} dt$$

$$+ \left(\frac{L_{12} I_1}{L}\right)^2 R \int_T^\infty e^{-\frac{2R(t-T)}{L}} dt$$

$$\simeq 2 \left(\frac{L_{12} I_1}{L}\right)^2 R \int_0^\infty dt\, e^{-\frac{2Rt}{L}} = \boxed{\frac{1}{L}(L_{12} I_1)^2}$$

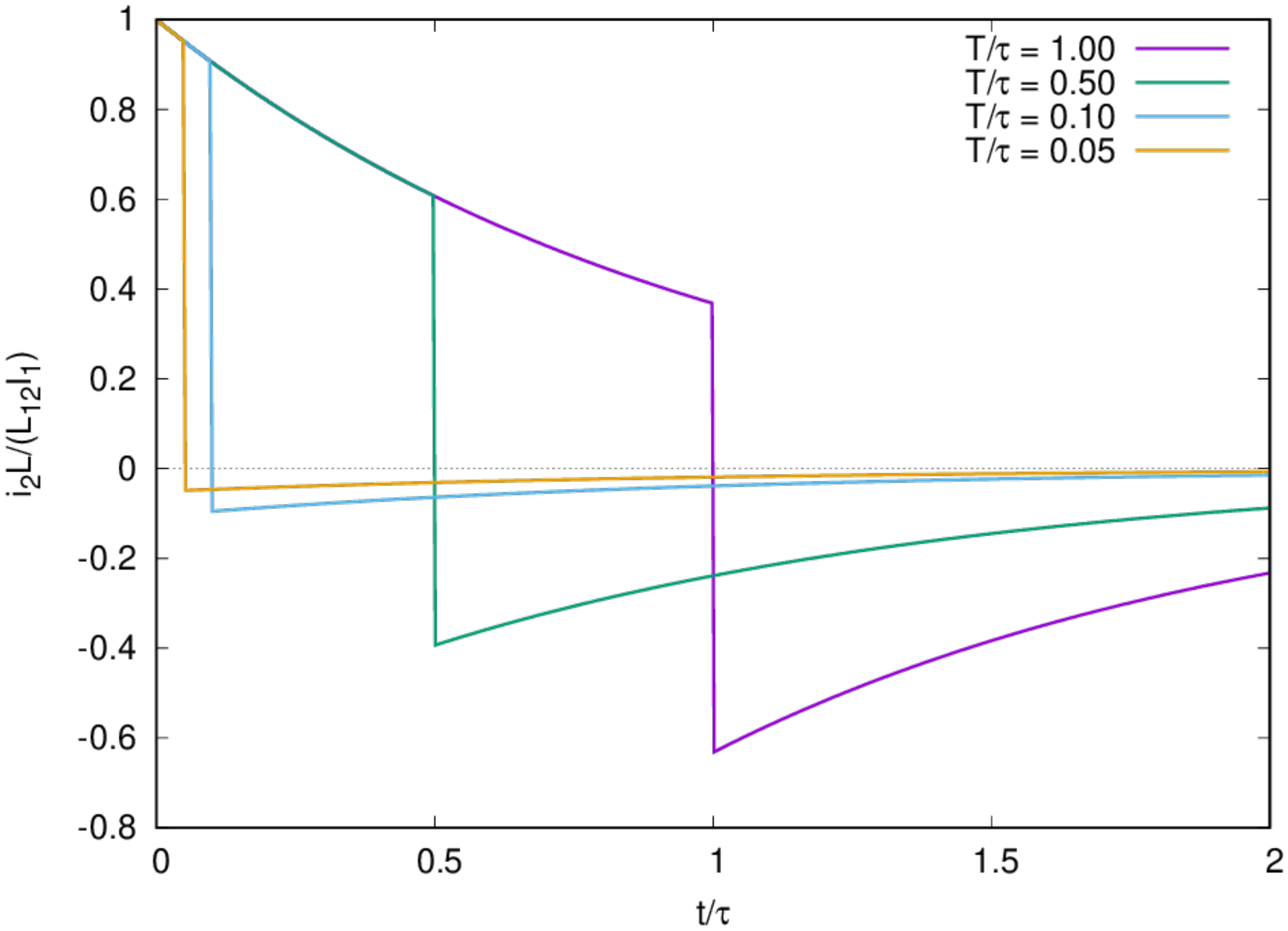
c) In order to graph the solution and check its properties, we need to find good scaling

$$\underbrace{\frac{i_2(t)\,L}{L_{12}\,I_1}}_{\text{dimensionless}} = \left\{ e^{-\frac{t}{\tau}} - \theta\left(\frac{t-T}{\tau}\right) e^{-\frac{(t-T)}{\tau}} \right\}, \quad \tau = \frac{L}{R} \text{ time scale}$$

So, how does this solution look for a shorter and shorter square pulse, which might be thought of like a delta-function limit if we take care of

$$T \to 0 \qquad \text{with} \quad T I_1 = \text{Constant}$$
$$I_1 \to \infty$$

But as we scale the solution we do not have to worry about this last point



we see that the limiting form for the solution as the pulse gets shorter is an individual square pulse. This can be found analytically, but then a special care has to be taken with respect to the derivative of the Dirac-delta-function