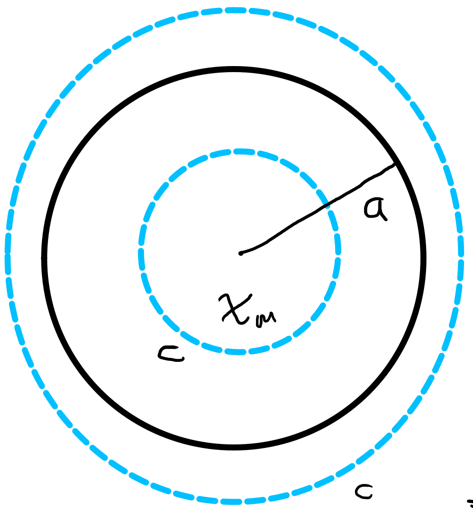
Problem 1



Current I flows in a long wire with radius a

Imagine two concentric circular paths for Amperes law

inside and outsidw the wire

$$\boxed{\oint_c \bar{H} \cdot d\bar{l} = J_f} \quad (1)$$

Uniform current density
angular symmetry
for $\bar{B}$ and $\bar{H}$

J and $\bar{A}$ only have
a z - component

(1) : $J_f^{enc} = \dfrac{\pi r^2}{\pi a^2} J_f = J_f \left(\dfrac{r}{a}\right)^2$

$\underline{r \leq a}$

$$2\pi r H_i = J_f \left(\frac{r}{a}\right)^2 \rightarrow H_i(r) = J_f \frac{r}{2\pi a^2}$$

$$\rightarrow \boxed{\bar{H}_i(r) = \hat{a}_\phi J_f \frac{r}{2\pi a^2}}$$

$\underline{r \geq a}$:

1 $\quad 2\pi r H_o = J_f$

$$\rightarrow \boxed{\bar{H}_o = \hat{a}_\phi \frac{J_f}{2\pi r}}$$

Boundary conditions

no surface current density

$$\hat{a}_{n_2} \times (\bar{H}_1 - \bar{H}_2) = \underbrace{\bar{J}_s = 0}$$

$$\rightarrow \hat{a}_r \times (\bar{H}_i - \bar{H}_o) = 0$$

$\rightarrow$ H is continuous in the surface of the wire

Linear material $\qquad \bar{H} = \frac{1}{\mu_o}\bar{B} - \bar{M}$

$\underline{r \geq a} \rightarrow \underline{\bar{M}_o = 0} \rightarrow \boxed{\bar{B}_o = \mu_o \bar{H} = \hat{a}_\phi \frac{J_f \mu_o}{2\pi r}}$

$r \leq a$ if linear

$$\bar{B} = \mu_0(\bar{H} + \bar{M}) = \mu_0(1+\chi_m)\bar{H}, \quad \text{as } \bar{M} = \chi_m \bar{H}$$

$$\boxed{\bar{B}_i = \hat{a}_\phi \mu_0 (1+\chi_m)\frac{J_f r}{2\pi a^2}, \quad \bar{M} = \chi_m \bar{H}_i = \hat{a}_\phi \mu_0 \chi_m \frac{J_f r}{2\pi a^2}}$$

Bound currents

In the bulk:

$$\bar{J}_B = \bar{\nabla} \times \bar{M} = \hat{a}_z \frac{1}{r}\frac{\partial}{\partial r}(r M_\phi)$$

$$= \underline{\hat{a}_z \, 2\mu_0 \chi_m \frac{J_f}{2\pi a^2}} \quad : \text{Constant}$$

On the surface

$$\bar{K}_B = \bar{M} \times \hat{n} = -\hat{a}_z \mu_0 \chi_m \frac{J_f}{a}$$

Total

$$\oint \bar{J}_B \cdot d\bar{s} + \oint_c \bar{K}_m \cdot d\bar{l}$$

$$= \hat{a}_z \cdot \left\{ \pi a^2 \cdot 2\mu_0 \chi_m \frac{J_f}{2\pi a^2} - 2\pi a \frac{\mu_0 \chi}{a} \right\} = \underline{0}$$

Problem 2

Sphere of radius a : $\overline{M} = M_0 \hat{a}_z \left(\frac{r}{a}\right)^2$

$$\overline{M} = M_0 \left\{ \hat{a}_r \cos\theta - \hat{a}_\theta \sin\theta \right\} \left(\frac{r}{a}\right)^2$$

d) Bound current density in the bulk

$$\boxed{\overline{J}_b} = \overline{\nabla} \times \overline{M} = \hat{a}_\phi \frac{1}{r} \left\{ \frac{\partial}{\partial r}(r M_\theta) - \frac{\partial M_r}{\partial \theta} \right\}$$

$$= \hat{a}_\phi \frac{M_0}{a^2} \left[-3r^2 \sin\theta + r^2 \sin\theta \right]$$

$$= \boxed{-2 \hat{a}_\phi M_0 \left(\frac{r}{a}\right)^2 \sin\theta}$$

c) Bound surface current density

$$\boxed{\overline{K}_b = \overline{M} \times \hat{a}_r = \hat{a}_\phi M_0 \sin\theta}$$

a) Equivalent magnetic charge density

$$\rho_m = -\bar{\nabla}\cdot\bar{M} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 M_r\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, M_\theta\right)$$

$$= -\frac{1}{r^2}\frac{M_o}{a^2}\partial_r\, r^4\cos\theta - \frac{M_o}{r\sin\theta}\partial_\theta\left(\frac{r^2}{a^2}\sin^2\theta\right)$$

$$= -\frac{3M_o}{a^2} r\cos\theta + \frac{2M_o}{a^2} r\cos\theta$$

$$= -\frac{M_o}{a^2} r\cos\theta$$

b) Equivalent magnetic surface charge

$$\sigma_m = \hat{n}\cdot\bar{M}\Big|_{r=a} = M_o\cos\theta$$

e) Find the total magnetic momentum of the sphere

Here, we have to be careful. we know from the expression for M that it must be in the z-direction

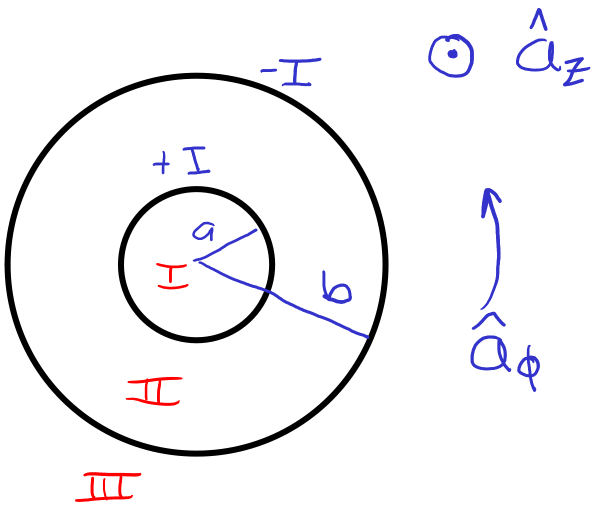
$$\overline{M} = M_o \hat{a}_z \left(\frac{r}{a}\right)^2$$

$$\rightarrow \quad \overline{m} = \hat{a}_z M_o 2\pi \int_0^a r^2 dr \int_0^\pi d\theta \sin\theta \frac{r^2}{a^2}$$

$$= \hat{a}_z M_o 2\pi \frac{a^5}{a^2 5} \left(-\cos\theta \Big|_0^\pi \right)$$

$$= \hat{a}_z \frac{4\pi}{5} M_o a^3$$

Problem 3



$$\oint \bar{H} \cdot d\bar{l} = J_t \quad (1)$$

II $a \leq r \leq b$:

(1) $\rightarrow 2\pi r H = I \rightarrow$ $\boxed{\bar{H}_{II} = \hat{a}_\phi \dfrac{I}{2\pi r}}$

III $r \geq b$

(1) $\rightarrow 2\pi r H = I - I = 0 \rightarrow$ $\boxed{\bar{H}_{III} = 0}$

I $r < a$

$$\boxed{\bar{H}_I = 0}$$

we have surface current densities, they through

$$\hat{a}_{n_2} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s$$

exactly determine the jumps in H at the two boundaries

$$\overline{H} = \frac{1}{\mu_0}\overline{B} - \overline{M} \longrightarrow \overline{B} = \mu_0(\overline{H} + \overline{M})$$

$$= \mu_0(1 + \chi_m)\overline{H} \quad \text{as} \quad \overline{M} = \chi_m \overline{H}$$

I $\quad \overline{B} = 0$

III $\quad \overline{B} = 0$

II $\quad \overline{B} = \mu_0(1 + \chi_m)\overline{H} = \hat{a}_\phi \dfrac{\mu_0(1 + \chi_m) I}{2\pi r}$

$$\overline{M} = \chi_m \overline{H} = \hat{a}_\phi \frac{\chi_m I}{2\pi r}$$