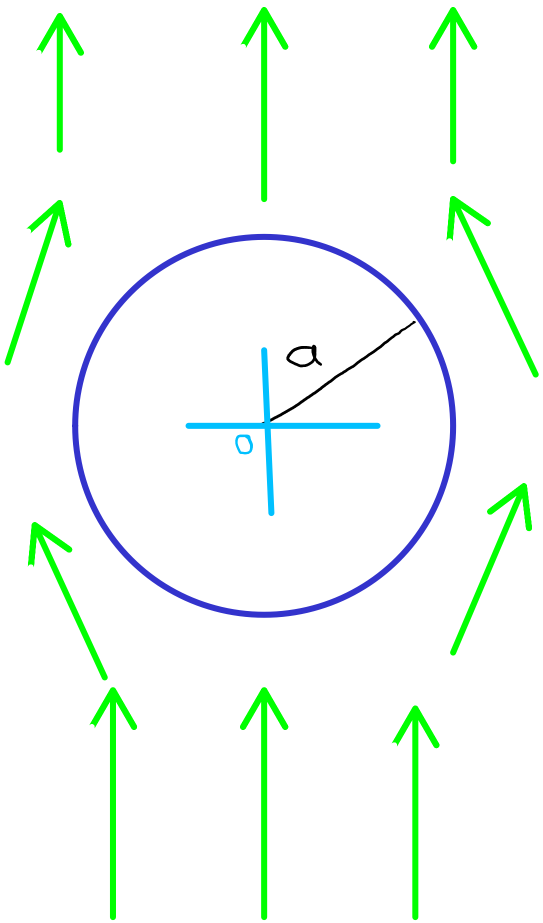
## Problem 1




Flow of current around a cylindrical hole drilled into a slab of material. Steady state, so the equation of continuity gives

$$\bar{\nabla} \cdot \bar{J} = 0$$

From Ohmes Law $\bar{J} = \sigma \bar{E}$ and $\bar{\nabla} \times \bar{E} = 0$ we get

$$\bar{\nabla} \times \left( \frac{\bar{J}}{\sigma} \right) = 0$$

so, we invent a potential $\phi$

$$\nabla^2 \phi = 0 , \qquad \bar{J} = -\bar{\nabla} \phi$$

with a "different" kind of boundary condition

$$\partial_r \phi (r,\phi) \Big|_{r=a} = 0 \qquad (*)$$

we need an additional boundary condition maintaining the steady current.

Far away from o, or the cylinder, we need the solution to be

$$\bar{J} = J_0 \hat{a}_x \quad (**) \quad \longrightarrow \quad \psi = -J_0 r \cos\phi \, , \quad \text{for} \quad r >> a$$

The general solution is

$$\psi_n(r,\phi) = r^n \left\{ A_n \sin(n\phi) + B_n \cos(n\phi) \right] + r^{-n} \left\{ A_n' \sin(n\phi) + B_n' \cos(n\phi) \right]$$

To satisfy (**) we need

$A_n = 0$ for all $n$ : mirror symmetry around x-axis

$B_n = 0$ if $n \neq 1$

if $n \neq 0$

$A_n' = 0$ for all $n$

$$\rightarrow \psi(r,\phi) = -J_0 r \cos\phi + \sum_{n=1}^{\infty} B_n' r^{-n} \cos(n\phi)$$

Now use (*)

$$-\partial_r \psi(r,\phi) = J_0 \cos\phi - \sum_{n=1}^{\infty} B_n' (-n) r^{-(n+1)} \cos(n\phi)$$

$$= 0$$

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For $r=a$ we need all $B'_n = 0$ if $n \neq 1$

$$\rightarrow J_0 \cos\phi + B'_1 \frac{\cos\phi}{a^2} = 0$$

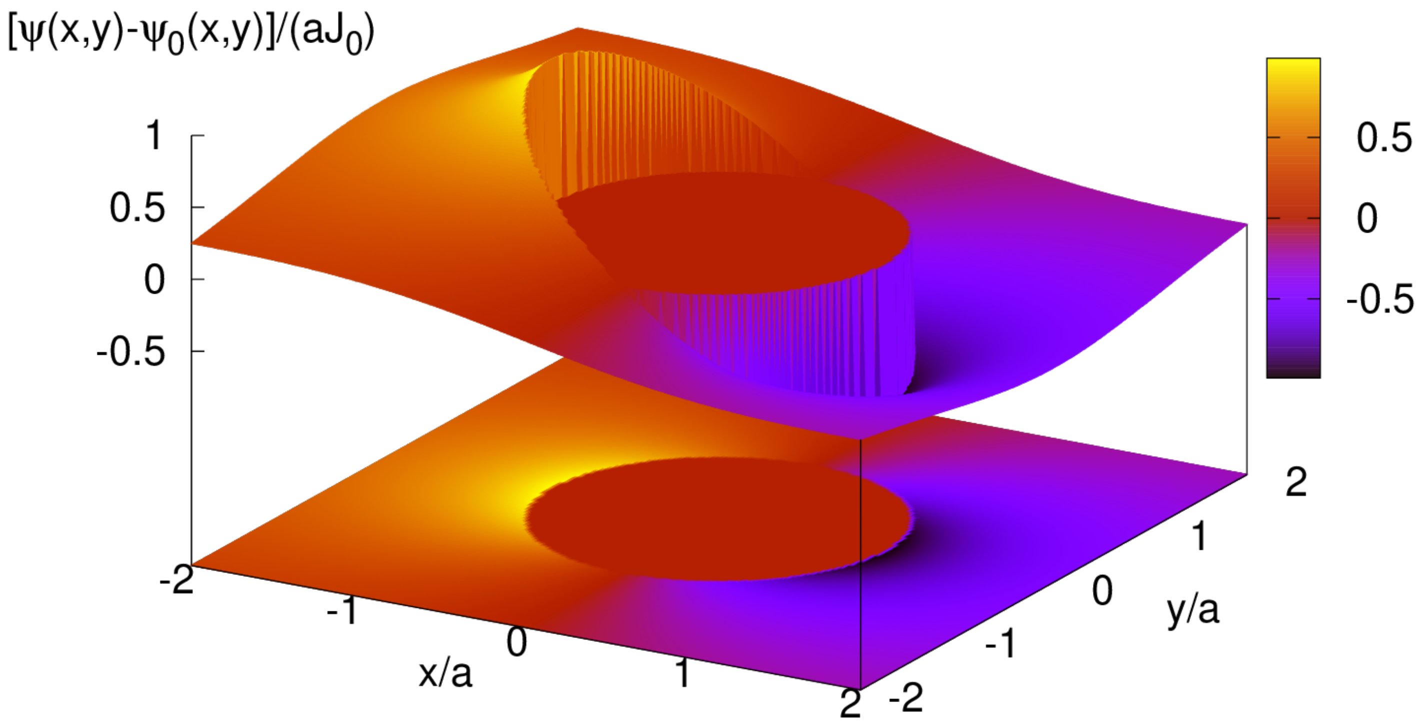
$$\rightarrow B'_1 = -J_0 a^2$$

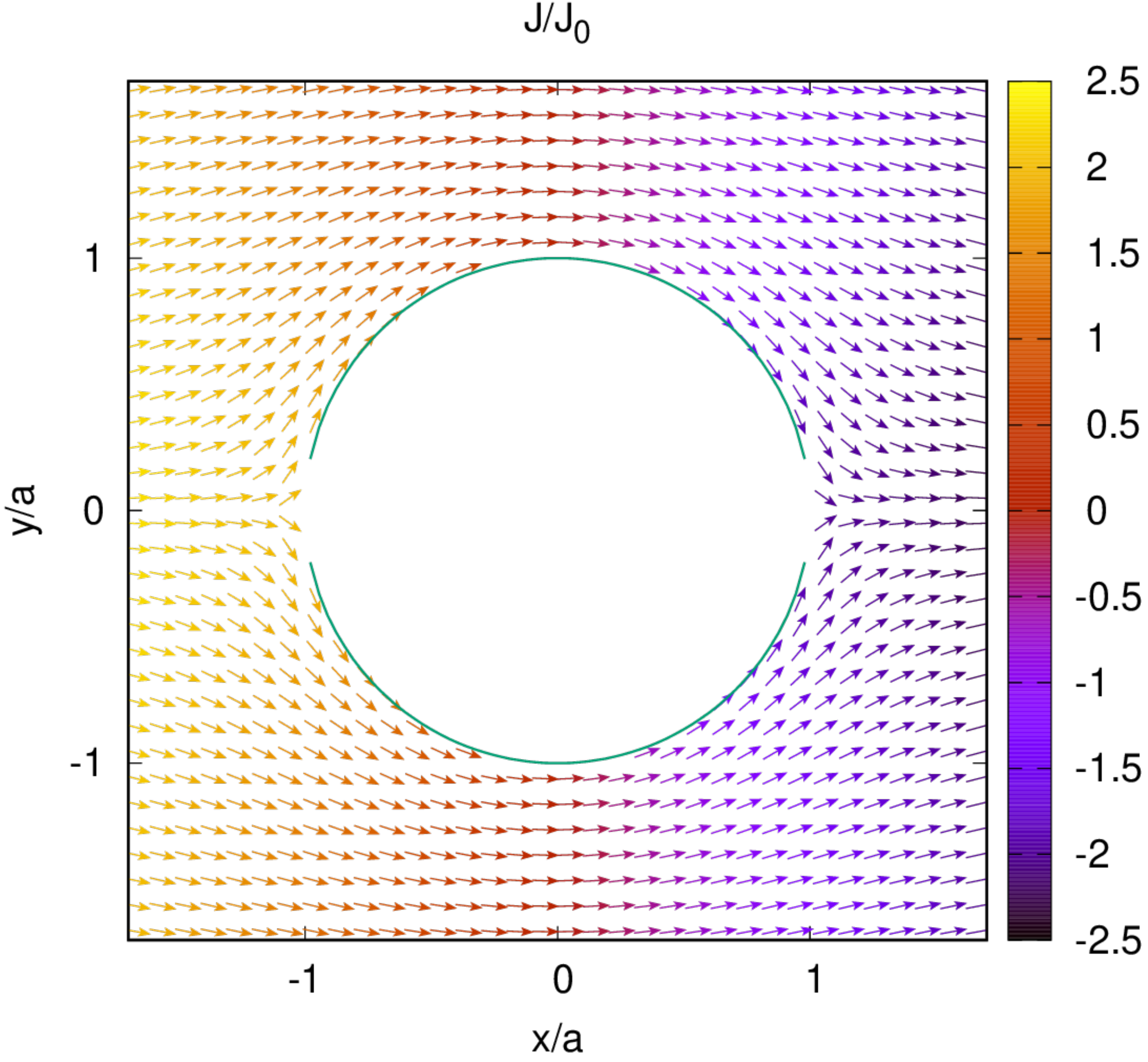
$$\rightarrow \boxed{\begin{aligned} \psi(r,\phi) &= -J_0 r \cos\phi - J_0 a^2 \frac{1}{r} \cos\phi \\ &= -J_0 \cos\phi \cdot \left\{ r + \frac{a^2}{r} \right\} \\ &= -J_0 r \cos\phi \cdot \left\{ 1 + \left(\frac{a}{r}\right)^2 \right\} \end{aligned}}$$

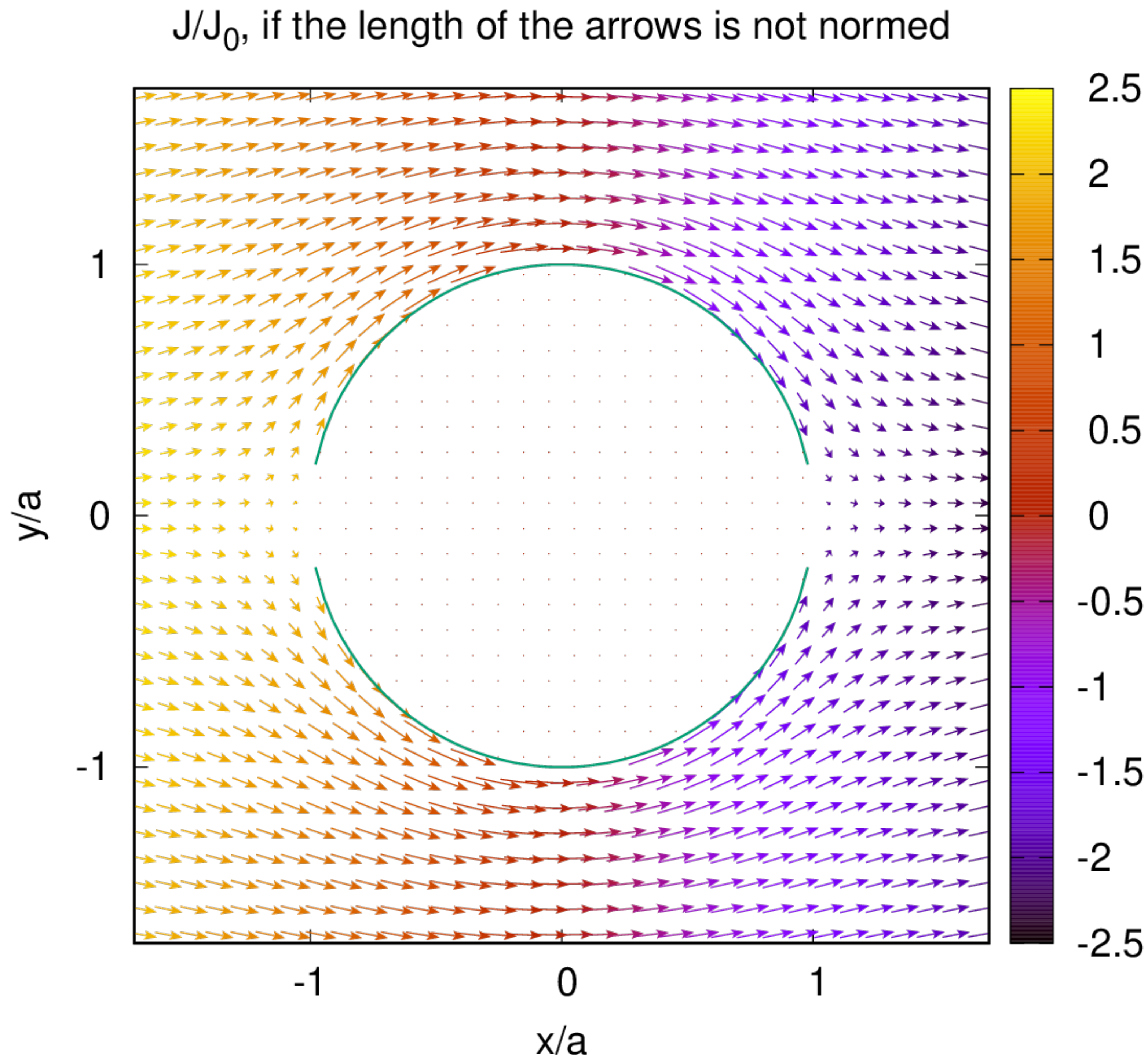
$$\boxed{\begin{aligned} \bar{J} = -\bar{\nabla}\psi &= -\hat{a}_r \partial_r \psi - \hat{a}_\phi \frac{1}{r} \partial_\phi \psi \\ &= +\hat{a}_r \left\{ J_0 \cos\phi \left( 1 - \left(\frac{a}{r}\right)^2 \right) \right\} \\ &\quad - \hat{a}_\phi \left\{ J_0 \sin\phi \left( 1 + \left(\frac{a}{r}\right)^2 \right) \right\} \end{aligned}}$$

Easiest is to plot the potential (a scalar function), remember $x = r\cos\phi$ and its variation is best seen by plotting $\left[\psi(x,y) - \psi_0(x,y)\right]/(J_0 a)$

where $\psi_0 = -J_0 x = -J_0 \cos\phi$




J/J$_0$
y/a
x/a
1
0
-1
-1
0
1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5


J/J0, if the length of the arrows is not normed
y/a
x/a
1
0
-1
-1
0
1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5