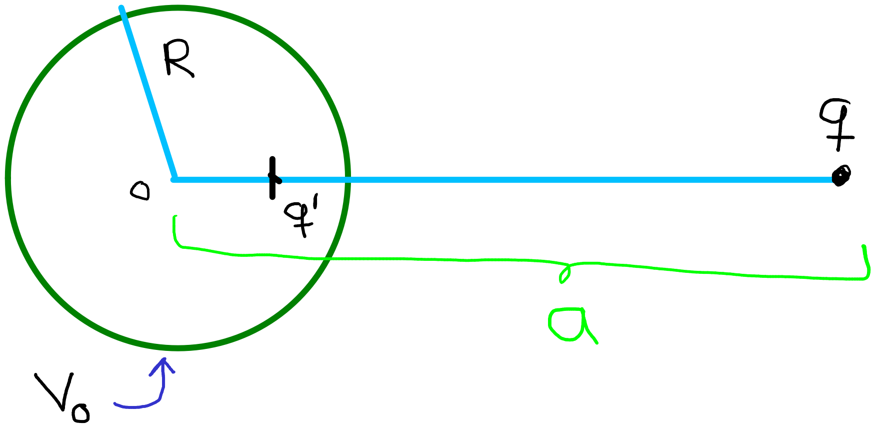
Problem 01



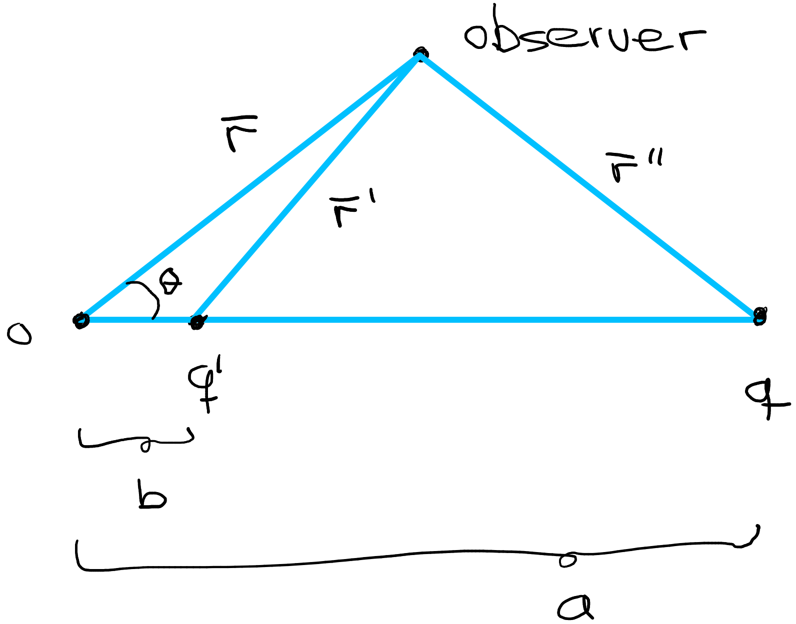
solid metal sphere at voltage V_0

method of images, for a grounded sphere

the book suggests a charge q' at b

$$q' = -\frac{R}{a} q$$

$$b = \frac{R^2}{a}$$



First, we check this location of q'

$$r' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$$

$$r'' = \sqrt{r^2 + a^2 - 2ra\cos\theta}$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{r''} + \frac{q'}{r'}\right\}$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{Rq}{\sqrt{r^2+\frac{R^4}{a^2}-\frac{2rR^2}{a}\cos\theta}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+\frac{r^2a^2}{R^2}-2ra\cos\theta}}\right]$$

$$= 0 \quad \text{if} \quad r = R$$

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we need the sphere at V_0 --> $\underbrace{\text{add a charge}}_{Q}$ to the middle of the sphere (image c.)

$$V_0 = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$$

Then we see from the book that the force between the sphere and q is

$$F = \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{Qq}{a^2}\right]$$

$$F = \frac{1}{4\pi\epsilon_0}\left[-\frac{q^2 R}{a(a^2-R^2)^2} + \frac{4\pi\epsilon_0 V_0 R q}{a^2}\right]$$

$$= -\frac{1}{4\pi\epsilon_0 a}\frac{q^2 R}{(a^2-R^2)^2} + q V_0 \frac{R}{a^2}$$

Correct dimensions! only original variables in the answer. No information from the images visible. Attractive force between a grounded sphere and a charge q, but an addition to the force comes from the finite potential. If $q > 0$ then the additional force is repulsinve if $V_0 > 0$

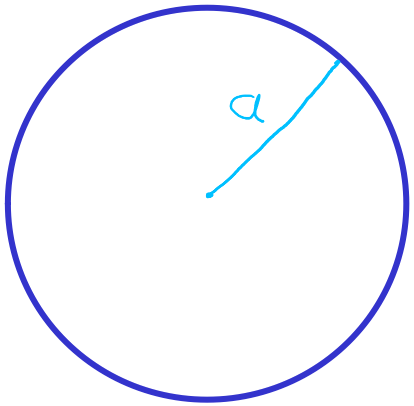
Positive charge q is not actracted to positive potential.

The first term is independent of the sign of q

Problem 02

Insulating cylindrical shell with surface charge density

$$\sigma(a,\phi) = \sigma_0 \sin(4\phi)$$



In cylindrical coordinates the general solution to the Poisson equation (no line charges):

$$V_n(r,\phi) = r^n\left\{A_n \sin(n\phi) + B_n \cos(n\phi)\right\} + r^{-n}\left\{A_n' \sin(n\phi) + B_n' \cos(n\phi)\right\}$$

The boundary condition is

$$\hat{a}_{n2}\cdot\left(\bar{D}_1 - \bar{D}_2\right) = \sigma \,, \quad \text{and} \quad \bar{D} = \epsilon_0 \bar{E}, \quad \bar{E} = -\bar{\nabla} V$$

$$\rightarrow \left\{\partial_r V^o(r,\phi) - \partial_r V^i(r,\phi)\right\} = -\frac{\sigma_0}{\epsilon_0}\sin(4\phi)$$

o : outer

i : inside

$$\rightarrow \quad V^{o}(r,\phi) = \sum_{n=1}^{\infty} \frac{A_n'}{r^n} \sin(n\phi)$$

$$V^{i}(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \sin(n\phi)$$

we need to use the boundary condition (*) and remember that sin(nφ) are orthogonal

$$\rightarrow \quad \left[-4\frac{A'}{r^5} \sin(4\phi) - 4 A r^3 \sin(4\phi) \right]_{r=a} = -\frac{\sigma_o}{\epsilon_o} \sin(4\phi)$$

$$\rightarrow \quad -4\frac{A'}{a^5} - 4 A a^3 = -\frac{\sigma_o}{\epsilon_o}$$

We also have $V^{o}(a,\phi) = V^{i}(a,\phi)$

$$\rightarrow \quad \frac{A'}{a^4} = A a^4$$

Taken together:

$$A' = A\, a^8$$

$$-4 A a^3 - 4 A a^3 = -\frac{\sigma_0}{\epsilon_0} \quad \rightarrow \quad A = \frac{\sigma_0}{8 a^3 \epsilon_0}$$

$$\text{and} \quad A' = \frac{\sigma_0}{8\epsilon_0} \cdot a^5$$

giving the solution

$$\boxed{\begin{aligned} V^o(r,\phi) &= \frac{\sigma_0 a}{8\epsilon_0}\left(\frac{a}{r}\right)^4 \sin(4\phi) \qquad r > a \\ V^i(r,\phi) &= \frac{\sigma_0 a}{8\epsilon_0}\left(\frac{r}{a}\right)^4 \sin(4\phi) \qquad r < a \end{aligned}}$$

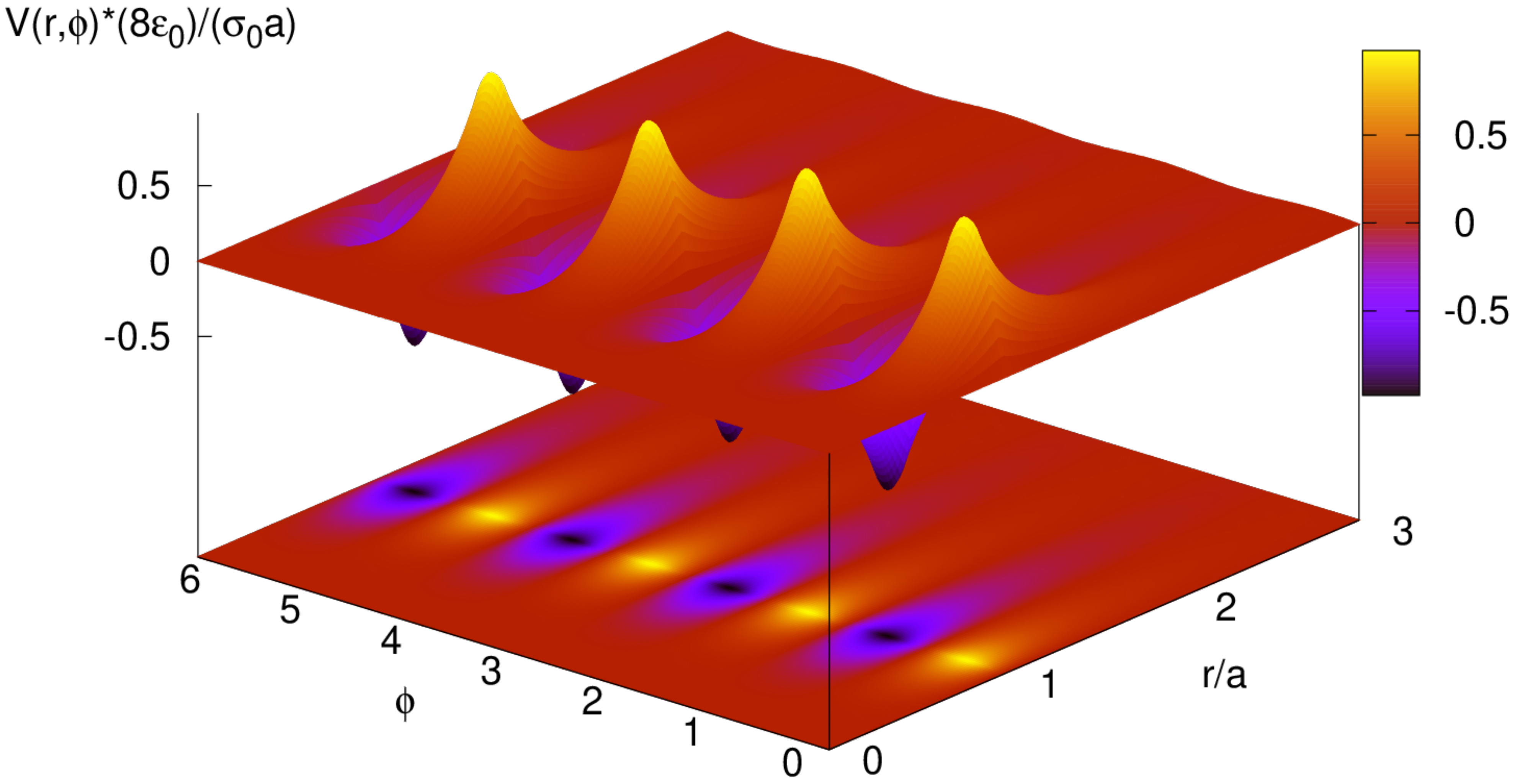
$$\overline{E} = -\overline{\nabla} V = -\hat{a}_r \partial_r V - \frac{\hat{a}_\phi}{r} \partial_\phi V$$

$$\bar{E}^{i} = -\hat{a}_r \frac{V_0}{2\epsilon_0}\left(\frac{r}{a}\right)^3 \sin(4\phi) - \hat{a}_\phi \frac{V_0}{2\epsilon_0}\left(\frac{r}{a}\right)^3 \cos(4\phi) \qquad r<a$$

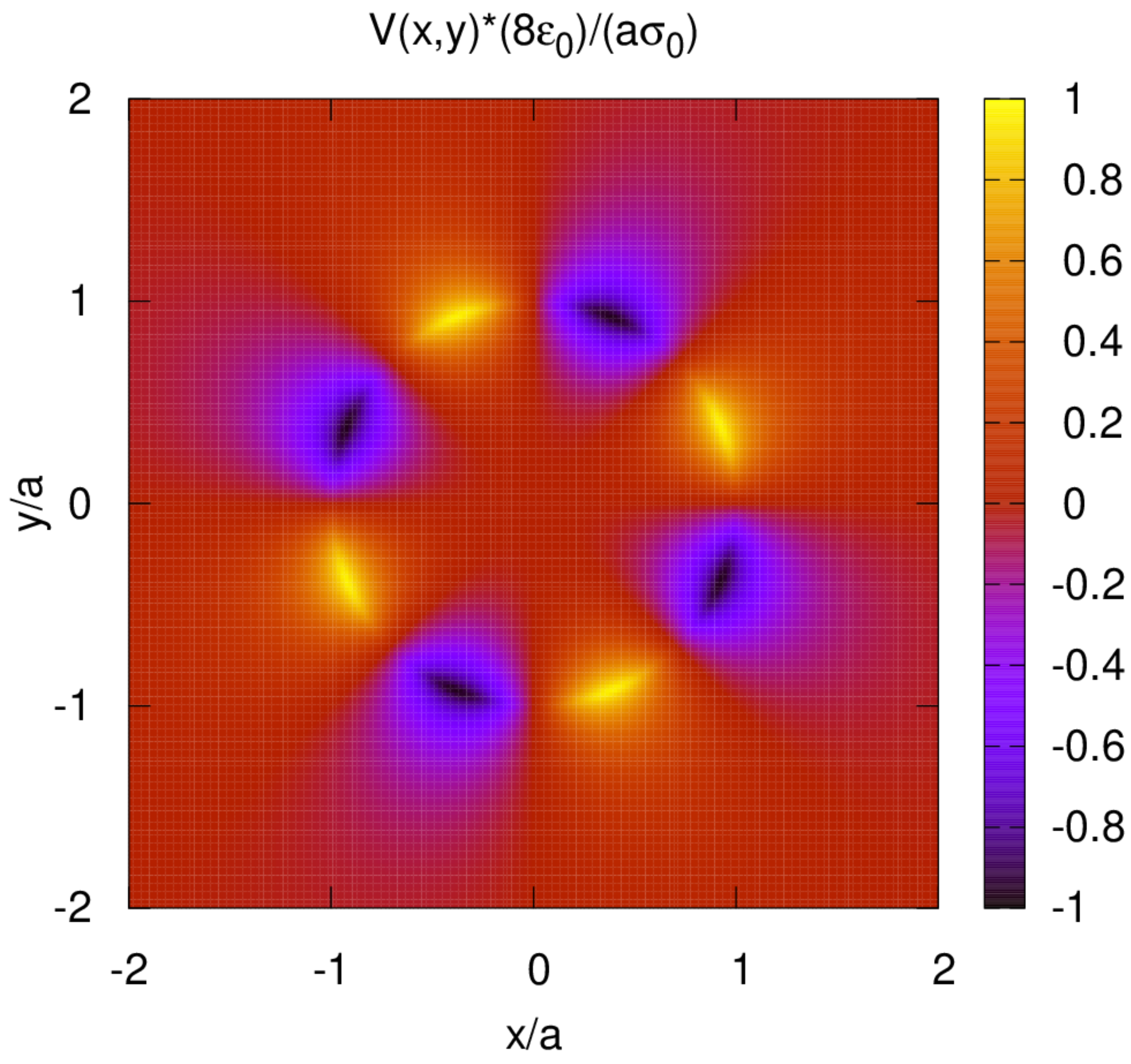
$$\bar{E}^{o} = \hat{a}_r \frac{V_0}{2\epsilon_0}\left(\frac{a}{r}\right)^5 \sin(4\phi) - \hat{a}_\phi \frac{V_0}{2\epsilon_0}\left(\frac{a}{r}\right)^5 \cos(4\phi) \qquad r>a$$

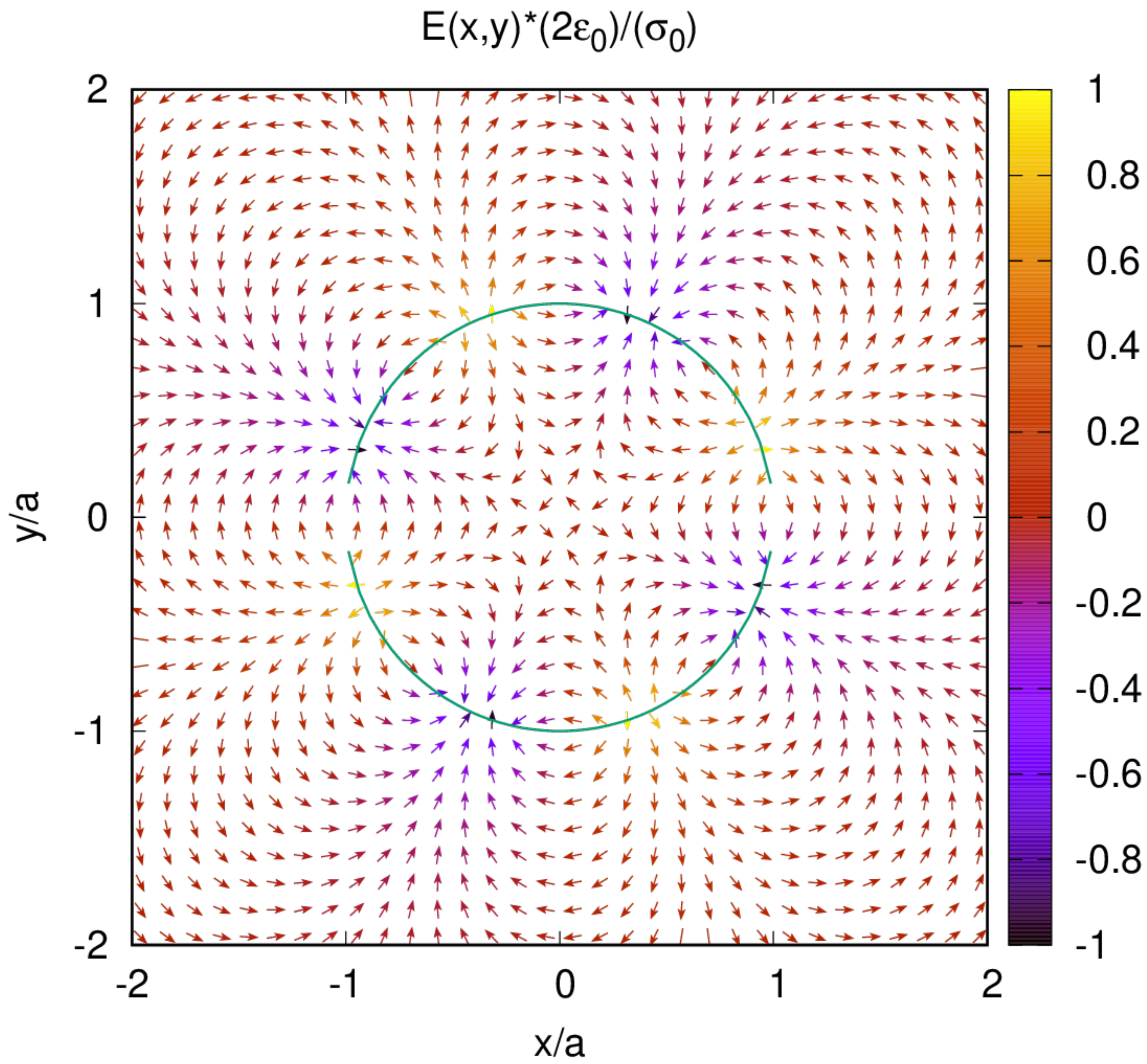
(c) The potential is a pure octopole potential as the system in whole is uncharged. An unbalanced charge would always create a $\ln(r)$ term in cylinder symmetry.

(b) I add pictures of the potential and the electric field on the next pages.



$V(r,\phi)*(8\varepsilon_0)/(\sigma_0 a)$
0.5
0
-0.5
0.5
0
-0.5
6
5
4
3
2
1
0
ϕ
0
1
2
3
r/a


V(x,y)*(8ε0)/(aσ0)
2
1
0
-1
-2
y/a
-2
-1
0
1
2
x/a
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1


E(x,y)*(2ε0)/(σ0)
y/a
x/a
2
1
0
-1
-2
2
1
0
-1
-2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1