1

1. Dæmaskammtur

Bolta hent beint upp með ferð v_0 í föstu þyngdarsviði. Hversu langt upp fer hann. Viðnámskraftur $f \sim v^2$

Hreyfijafnan

$$m\ddot{z} = -mDv^2 - mg$$

Vegna viðnámsliðar er heppilegt að umrita fyrir v í stað z

$$\ddot{z} = \frac{d^2z}{dt^2} = \frac{d}{dt}\left\{\frac{dz}{dt}\right\} = \frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz}$$

2

Hreyfijafnan er því

$$mv\frac{dv}{dz} = -mDv^2 - mg$$

$$\to \frac{v\,dv}{Dv^2+g} = -dz$$

$$\int_{v_0}^{v}\frac{v'dv'}{D(v')^2+g} = -\int_0^z dz' \to \frac{1}{2D}\ln(D(v')^2-g)\Big|_{v_0}^{v} = -z$$

$$\to z = -\frac{1}{2D}\ln\left\{\frac{Dv^2+g}{Dv_0^2+g}\right\} = \frac{1}{2D}\ln\left\{\frac{Dv_0^2+g}{Dv^2+g}\right\}$$

Hæsti punkturinn er þegar $v = 0$

$$\to z_{max} = \frac{1}{2D}\ln\left\{\frac{Dv_0^2+g}{g}\right\}$$

3

$$z_{max} = \frac{1}{2D}\ln\left\{\frac{Dv_0^2}{g}+1\right\}$$

$$= \frac{1}{2D}\left\{\frac{Dv_0^2}{g} - \frac{(Dv_0^2)^2}{2g^2} + o(D^3)\right\}$$

$$\to z_{max}(D\to 0) = \frac{v_0^2}{2g}$$

best er graf af

$$\frac{2z_m g}{v_0^2} = \frac{g}{Dv_0^2}\ln\left\{\frac{Dv_0^2}{g}+1\right\}$$

$$[D] = \frac{1}{L}$$

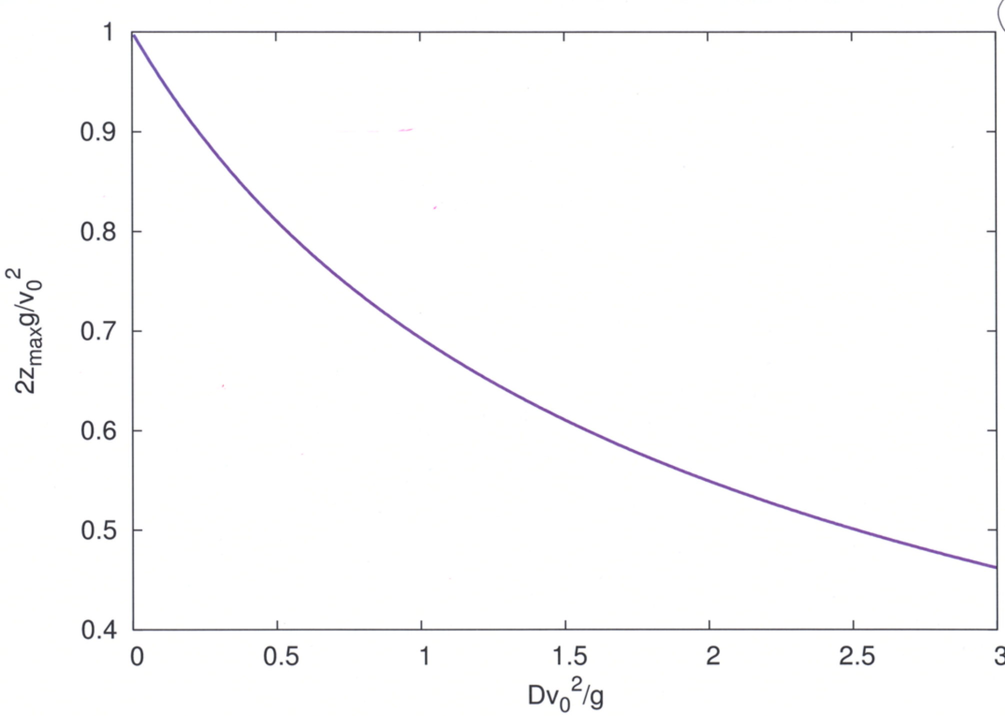
$$\to \left[\frac{Dv_0^2}{g}\right] = 1$$

og $\frac{2z_m g}{v_0^2}(D\to 0) = 1$

(2) Kraftur á ögn í kúluhnitum $F(r,\theta,\phi)$
Hvernig er hreyfijafnan?

Vandinn hér er að $\hat{r}, \hat{\theta}$ og $\hat{\phi}$ eru tímaháðir, það á ekki við fyrir Kartíst hnit. Sjá (1.14) í bók

3b



Staðsetning agnarinnar er $\bar r = r\hat e_r$ eða $r\hat r$
því er hreyfijafna hennar

$$m\ddot{\bar r} = \bar F$$

$$\bar v = \dot{\bar r} = \dot r \hat e_r + r\dot{\hat e}_r$$

Vitum að einingarvigrarnir í Kartískum hnitum eru óháðir tíma → leiðum

$$\hat e_r = \hat e_1 \sin\theta\cos\phi + \hat e_2 \sin\theta\sin\phi + \hat e_3\cos\theta$$

$$\hat e_\theta = \hat e_1 \cos\theta\cos\phi + \hat e_2\cos\theta\sin\phi - \hat e_3\sin\theta$$

$$\hat e_\phi = -\hat e_1\sin\phi + \hat e_2\cos\phi$$

$$\to \dot{\hat e}_r = \hat e_1\left\{\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi\right\} + \hat e_2\left\{\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi\right\} - \hat e_3\left\{\dot\theta\sin\theta\right\}$$

$$\to \dot{\hat e}_r = \hat e_\theta\dot\theta + \hat e_\phi\dot\phi\sin\theta$$

eins fæst

$$\dot{\hat e}_\theta = -\hat e_r\dot\theta + \hat e_\phi\dot\phi\cos\theta$$

$$\dot{\hat e}_\phi = -\hat e_r\dot\phi\sin\theta - \hat e_\theta\dot\phi\cos\theta$$

$$\to \bar v = \dot{\bar r} = \dot r\hat e_r + r\dot{\hat e}_r = \dot r\hat e_r + r\hat e_\theta\dot\theta + r\hat e_\phi\dot\phi\sin\theta$$

$$\bar a = \dot{\bar v} = \ddot r\hat e_r + \dot r\dot{\hat e}_r + \dot r\hat e_\theta\dot\theta + r\dot{\hat e}_\theta\dot\theta + r\hat e_\theta\ddot\theta + \dot r\hat e_\phi\dot\phi\sin\theta + r\dot{\hat e}_\phi\dot\phi\sin\theta + r\hat e_\phi\dot\phi\dot\theta\cos\theta + r\hat e_\phi\ddot\phi\sin\theta$$

$$\bar a = \ddot r\hat e_r + \dot r\left\{\hat e_\theta\dot\theta + \hat e_\phi\dot\phi\sin\theta\right\} + \dot r\hat e_\theta\dot\theta$$
$$+ r\left\{-\hat e_r\dot\theta + \hat e_\phi\dot\phi\cos\theta\right\}\dot\theta + r\hat e_\theta\ddot\theta + \dot r\hat e_\phi\dot\phi\sin\theta$$
$$+ r\left\{-\hat e_r\dot\phi\sin\theta - \hat e_\theta\dot\phi\cos\theta\right\}\dot\phi\sin\theta + r\hat e_\phi\dot\phi\dot\theta\cos\theta + r\hat e_\phi\ddot\phi\sin\theta$$

$$\bar a = \hat e_r\left[\ddot r - r\dot\theta^2 - r\dot\phi^2\sin^2\theta\right] + \hat e_\theta\left[r\ddot\theta + 2\dot r\dot\theta - r\dot\phi^2\cos\theta\sin\theta\right]$$
$$+ \hat e_\phi\left\{r\ddot\phi\sin\theta + 2\dot r\dot\phi\sin\theta + 2r\dot\phi\dot\theta\cos\theta\right\}$$
$$= \frac{1}{m}\left\{\hat e_r F_r + \hat e_\theta F_\theta + \hat e_\phi F_\phi\right\}$$

(3) Bát ýtt af stað með v_0, $F = -F_0 e^{+\beta v}$ viðnámskr.

Hreyfijafna $ma = m\ddot x = m\dot v = -F_0 e^{+\beta v}$

$$\to m\frac{dv}{dt} + F_0 e^{+\beta v} = 0, \quad v(0) = v_0$$

Aðgreinanleg afleiðujafna

$$dt = -\frac{m}{F_0}e^{-\beta v}dv \to \int_0^t dt' = -\frac{m}{F_0}\int_{v_0}^{v(t)} dv\, e^{-\beta v}$$

$$\to t = +\frac{m}{F_0\beta}\left\{e^{-\beta v(t)} - e^{-\beta v_0}\right\}$$

$$+\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\} = +e^{-\beta v(t)} \to -\beta v(t) = \ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\}$$

⑧

$$v(t) = -\frac{1}{\beta}\ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\}$$

Finnum t_m þegar $v(t_m) = 0$

$$0 = -\frac{1}{\beta}\ln\left\{\frac{F_0\beta t_m}{m} + e^{-\beta v_0}\right\}$$

$$\ln(1) = 0 \quad \to \quad \frac{F_0\beta t_m}{m} + e^{-\beta v_0} = 1$$

eða

$$t_m = \frac{m}{F_0\beta}\left\{1 - e^{-\beta v_0}\right\}$$

max tíminn, en reynum eftir að finna vegalengdina

$$\frac{dx}{dt} = -\frac{1}{\beta}\ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\}$$

⑨

$$\int_0^x dx' = -\frac{1}{\beta}\int_0^t dt' \ln\left\{\frac{F_0\beta t'}{m} + e^{-\beta v_0}\right\}$$

notum

$$\int dx \ln(a+bx) = \left\{x + \frac{a}{b}\right\}\ln(a+bx) - x \qquad (GR: 2.729)$$

(eða maxima, en þá þarf að henda einum aukafasta)

$$x = -\frac{1}{\beta}\left\{\left\{t + \frac{e^{-\beta v_0} m}{F_0\beta}\right\}\ln\left\{e^{-\beta v_0} + \frac{F_0\beta t}{m}\right\} - t - \frac{e^{-\beta v_0} m}{F_0\beta}(-\beta v_0)\right\}$$

$$= -\frac{1}{\beta}\left\{\left\{t + \frac{e^{-\beta v_0} m}{F_0\beta}\right\}\ln\left\{e^{-\beta v_0} + \frac{F_0\beta t}{m}\right\} - t + \frac{e^{-\beta v_0} m}{F_0\beta}\beta v_0\right\}$$

⑩

Vegalengdin verður $x(t_m)$

$$x(t_m) = -\frac{1}{\beta}\left\{\frac{m}{F_0\beta}\underbrace{\ln\left[e^{-\beta v_0} + 1 - e^{-\beta v_0}\right]}_{=0} - \frac{m}{F_0\beta}\left[1 - e^{-\beta v_0}\right] + \frac{m}{F_0\beta}e^{-\beta v_0}\beta v_0\right\}$$

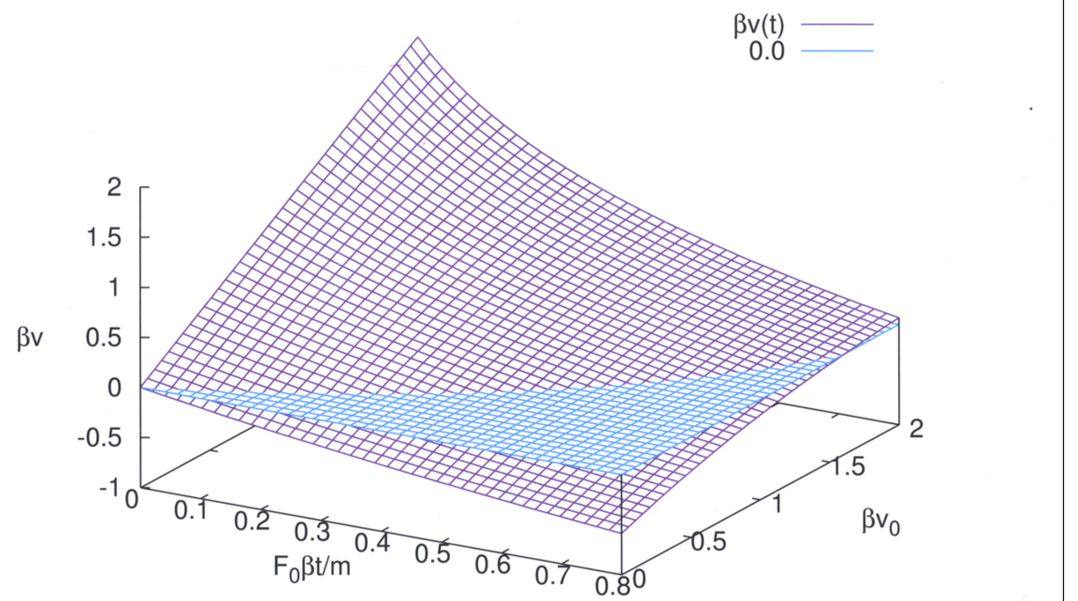
$$x(t_m) = +\frac{m}{\beta^2 F_0}\left\{1 - e^{-\beta v_0}(1 + \beta v_0)\right\}$$

Gröf

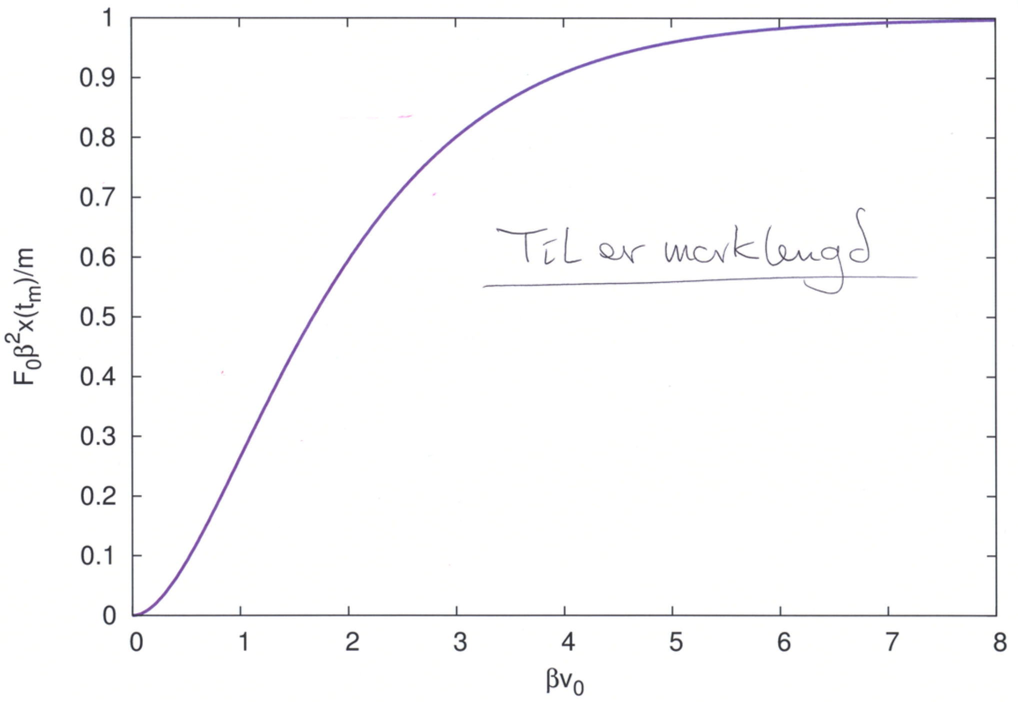
$$\beta v(t) = -\ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\} \quad \to \quad \beta v \text{ vs. } \frac{F_0\beta}{m}t$$

$$\frac{F_0\beta^2}{m}x(t_m) = \left\{1 - e^{-\beta v_0}(1 + \beta v_0)\right\} \quad \to \quad \frac{F_0\beta^2}{m}x \text{ vs. } \beta v_0$$

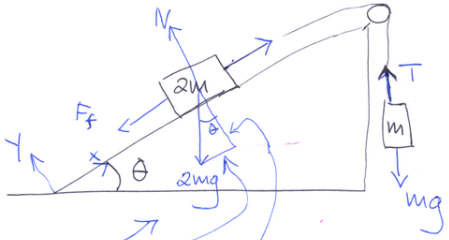
⑪



(12)



(4) (13)



fyrir hvaða θ fæst jafnur hraði fyrir massana

$$mg - T = ma$$

$$\rightarrow T = m(g-a)$$

y-hnit: $N - 2mg\cos\theta = m\ddot{y} = 0$

$$\rightarrow N = 2mg\cos\theta$$

x-hnit: $T - 2mg\sin\theta - 2\mu_k mg\cos\theta = ma$

Við leitum að θ_0 sem gefur $a = 0$

$$\rightarrow (g-a) - 2g\sin\theta_0 - 2\mu_k g\cos\theta_0 = 0$$

(14)

$$g - 2g\sin\theta_0 - 2\mu_k g\cos\theta_0 = 0$$

$$\frac{1}{2} - \sin\theta_0 - \mu_k\cos\theta_0 = 0$$

$$\frac{1}{2} - \sin\theta_0 - \mu_k\sqrt{1-\sin^2\theta} = 0$$

$$\frac{1}{2} - \sin\theta_0 = \mu_k\sqrt{1-\sin^2\theta_0} \rightarrow \left(\frac{1}{2} - \sin\theta_0\right)^2 = \mu_k^2\left(1-\sin^2\theta_0\right)$$

$$\rightarrow \sin^2\theta_0 + \frac{1}{4} - \sin\theta_0 = \mu_k^2\left(1-\sin^2\theta_0\right)$$

$$(1+\mu_k^2)\sin^2\theta_0 - \sin\theta_0 + \left(\frac{1}{4} - \mu_k^2\right) = 0$$

$$\rightarrow \sin\theta_0 = \frac{1 \pm \sqrt{1 - 4(1+\mu_k^2)(\frac{1}{4} - \mu_k^2)}}{2(1+\mu_k^2)} = \frac{1 \pm \mu_k\sqrt{3+4\mu_k^2}}{2(1+\mu_k^2)}$$

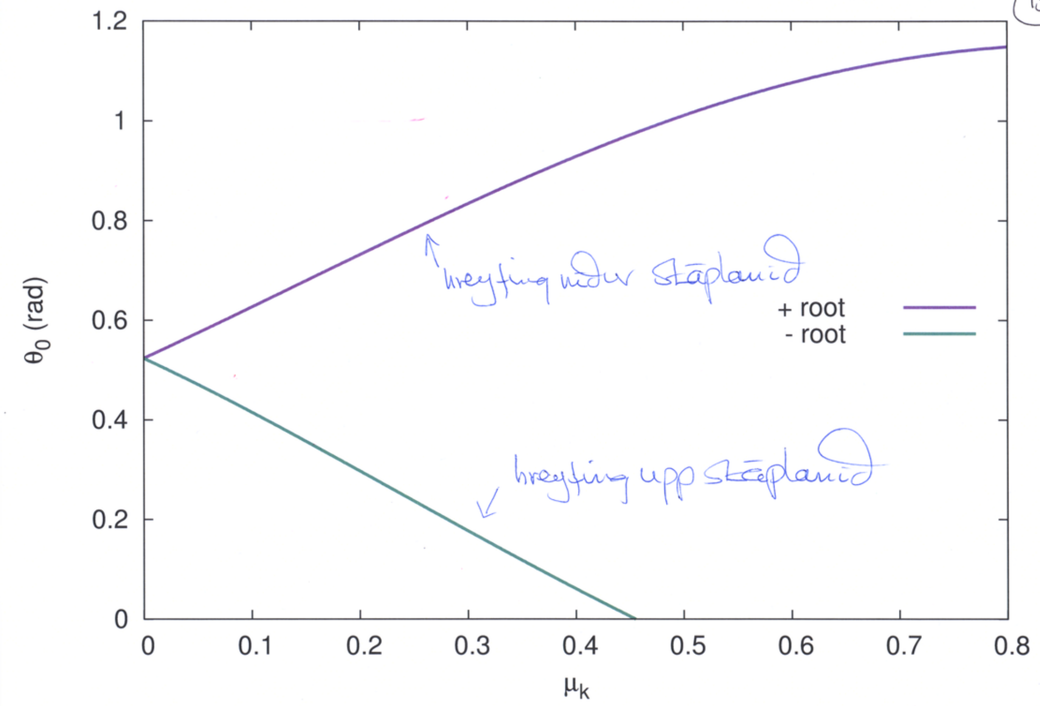
(15)

$$\theta_0 = \arcsin\left\{\frac{1 \pm \mu_k\sqrt{3+4\mu_k^2}}{2(1+\mu_k^2)}\right\}$$

tvær rætur

eini staðurinn þ.s. μ_k' kemur fyrir

hreyfing í tvær stefnur



(5) 2-21 í bók

Kasthreyfing í x-z sléttu

$$\left.\begin{aligned} x &= v_0 t \cos\theta \\ z &= v_0 t \sin\theta - \tfrac{1}{2} g t^2 \end{aligned}\right\} \quad \bar{r} = x\hat{e}_x + z\hat{e}_z = (v_0 t\cos\theta, 0, v_0 t \sin\theta - \tfrac{1}{2}gt^2)$$

$$\rightarrow \bar{p} = m\dot{\bar{r}} = m(v_0\cos\theta, 0, v_0\sin\theta - gt)$$

$$\bar{L} = \bar{r}\times\bar{p} = (v_0 t\cos\theta, 0, v_0 t\sin\theta - \tfrac{1}{2}gt^2)\, m \times (v_0\cos\theta, 0, v_0\sin\theta - gt)$$

$$= -\hat{e}_y\left\{v_0^2 t\cos\theta\sin\theta - v_0 t\cos\theta\, gt - v_0^2 t\sin\theta\cos\theta + v_0\cos\theta\,\tfrac{1}{2}gt^2\right\} m$$

$$= \hat{e}_y \tfrac{1}{2} m g v_0 t^2 \cos\theta$$

$$\rightarrow \dot{\bar{L}} = \hat{e}_y\, m g v_0 t \cos\theta$$

Kraftarinn á ögnina í kasthreyfingu er

$$\bar{F} = -mg\hat{e}_z$$

$$\rightarrow \bar{N} = \bar{r}\times\bar{F} = (v_0 t\cos\theta, 0, v_0 t\sin\theta - \tfrac{1}{2}gt^2) \times (0, 0, -mg)$$

$$= \hat{e}_y\left\{+ m g v_0 t\cos\theta\right\}$$

$$\rightarrow \dot{\bar{L}} = \frac{d\bar{L}}{dt} = \bar{N}$$

(6) Fall í föstu þyngdarsviði með viðnámskr. $\sim v^4$

Setjum $v_0 = 0$

z ↓

Hreyfijafna $\quad m\ddot{z} = -m\gamma v^4 + mg$

eins og í dæmi (1) $\quad m v \frac{dv}{dz} = -m\gamma v^4 + mg$

$$\rightarrow \frac{v\,dv}{-\gamma v^4 + g} = dz \quad \text{eða} \quad \frac{v\,dv}{\gamma v^4 - g} = -dz$$

Notum GR 2.132.2

$$\int \frac{x\,dx}{bx^4 + a} = \frac{1}{4i\sqrt{ab}} \ln\left\{\frac{a + x^2 i\sqrt{ab}}{a - x^2 i\sqrt{ab}}\right\} \qquad ab < 0$$

$$\int_0^z dz' = \int_0^v \frac{v'\,dv'}{-\gamma(v')^4 + g} \rightarrow z = \frac{1}{4\sqrt{\gamma g}} \ln\left\{\frac{g + v^2\sqrt{\gamma g}}{g - v^2\sqrt{\gamma g}}\right\}$$

Án viðnáms gildir

$$\ddot{z} = g \quad \rightarrow \quad \dot{z} = v = gt$$

og

$$\frac{v\,dv}{g} = dz \quad \rightarrow \quad \int_0^v v'dv' = g\int_0^z dz'$$

$$\rightarrow \quad \frac{1}{2}v^2 = gz \quad \rightarrow \quad z = \frac{1}{2}\frac{v^2}{g}$$

$$z = \frac{1}{4\sqrt{rg}} \ln\left\{\frac{1+v^2\sqrt{\frac{r}{g}}}{1-v^2\sqrt{\frac{r}{g}}}\right\} \approx \frac{1}{4\sqrt{rg}}\left[2v^2\sqrt{\frac{r}{g}} + \frac{2}{3}\left(\frac{r}{g}\right)^{3/2}v^6 + \dots\right.$$

$$\approx \frac{v^2}{2g} + \frac{v^6}{6}\frac{r}{g^2} + \dots\dots$$

$$F = m\frac{dv}{dt} = mg - m\gamma v^4$$

$$\rightarrow \quad \frac{dv}{-\gamma v^4 + g} = dt$$

Notum GR 2.132.1

$$\int \frac{dx}{bx^4 + a} = \frac{\alpha'}{4a}\left[\ln\left(\frac{x+\alpha'}{x-\alpha'}\right) + 2\arctan\left(\frac{x}{\alpha'}\right)\right]$$

$$\alpha' = \sqrt[4]{-\frac{a}{b}}$$

Því er hægt að leiða út

$$t = F(v)$$

① 3-13 í bók, Markdeyfðar sveifill ①

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad \to \omega_0^2 = \beta^2$$

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0 \quad \leftarrow \text{hreyfij.}$$

Reynum lausn á hann $x(t) = y(t)e^{-\beta t}$

$$\to \dot{x} = \dot{y}e^{-\beta t} - \beta y e^{-\beta t} = e^{-\beta t}\{\dot{y} - \beta y\}$$

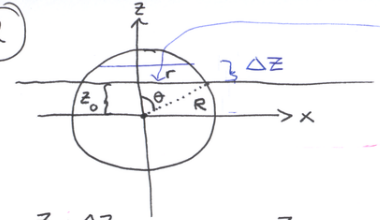
$$\ddot{x} = -\beta e^{-\beta t}\{\dot{y} - \beta y\} + e^{-\beta t}\{\ddot{y} - \beta\dot{y}\} = e^{-\beta t}\{\ddot{y} - 2\beta\dot{y} + \beta^2 y\}$$

Setjum inn í hreyfij.

$$e^{-\beta t}\{\ddot{y} - 2\beta\dot{y} + \beta^2 y + 2\beta\dot{y} - 2\beta^2 y + \beta^2 y\} = 0$$

$\to \ddot{y} = 0$ með lausn $y = A + Bt \to x(t) = \{A + Bt\}e^{-\beta t}$

② ②



Reiknum rúmmál sneiðar

$$r = R\sin\theta_0, \quad z_0 = R\cos\theta_0$$

$$r^2 = R^2 - z^2$$

$$\int_{z_0}^{z_0+\Delta z} \pi r^2 dz' = \int_{z_0}^{z} \pi(R^2 - z'^2)dz' = \pi\left(R^2 z' - \frac{z'^3}{3}\right)\Big|_{z_0}^{z} = V(z, z_0)$$

$$= \pi\left\{R^2(z - z_0) - \frac{z^3 - z_0^3}{3}\right\}$$

Hér er gott að samfæra sig um að $V(R,0)$ er $\frac{2\pi R^3}{3}$ ← hálf kúla

---

Á kúluna verkar þyngdarkraftur $-Mg = -\rho V_0 g$ og flotkraftur þegar kúlan marar með miðju z_0 undir yfirborði eru þeir í jafnvægi

③

Þegar ég ýti henni niður um Δz verkar flotkraftur vegna $V(z_0+\Delta z, z_0)$ beint upp

$$F_f(\Delta z) = \rho_0 V(z_0 + \Delta z, z_0) g = \rho_0 g\pi\left\{R^2\Delta z - \frac{(z_0+\Delta z)^3 - z_0^3}{3}\right\}$$

$$= \rho_0 g\pi\left\{\Delta z(R^2 - z_0^2) - (\Delta z)^2 z_0 - (\Delta z)^3\right\}$$

$$Ma = \rho V_0 (\ddot{\Delta z}) = -\rho_0 g\pi\left\{\Delta z(R^2 - z_0^2) - (\Delta z)^2 z_0 - (\Delta z)^3\right\}$$

$$\to (\ddot{\Delta z}) = -\frac{g\rho_0\pi}{\rho V_0}\left\{\Delta z(R^2 - z_0^2) - (\Delta z)^2 z_0 - (\Delta z)^3\right\}$$

④

Ef við tökum bara línulega liðinn?

$$(\ddot{\Delta z}) + \left\{\frac{g\rho_0\pi}{\rho V_0}(R^2 - z_0^2)\right\}\Delta z = 0$$

ef hún er í kafi $z_0 = R$ → engin sveifla

sveiflan er háð z_0

Annars er afleiðujafnan ólínuleg

Ef hún er hálf í kafi $z_0 = 0$

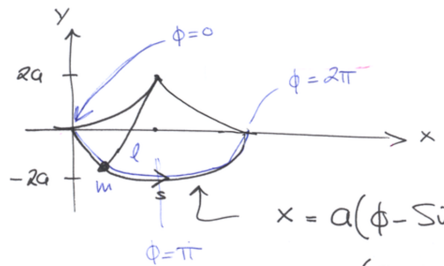
$$\to (\ddot{\Delta z}) + \left\{\frac{g\rho_0\pi R^2}{\rho V_0}\right\}\left\{\Delta z - \frac{1}{R^2}(\Delta z)^3\right\}$$

Jafnvel þá er sveiflan ósamhverf

(5)

(3) Hjólferilssveifill

Skilgreining ferils



$$x = a(\phi - \sin\phi) \quad \to \quad dx = a(1-\cos\phi)\,d\phi$$

$$y = a(\cos\phi - 1) \quad \to \quad dy = -a\sin\phi\,d\phi \qquad (*)$$

Eftir ferlinum

$$(ds)^2 = (dx)^2 + (dy)^2 = a^2\left\{(1-\cos\phi)^2 + \sin^2\phi\right\}(d\phi)^2$$

$$= 2a^2(1-\cos\phi)(d\phi)^2 = 4a^2\sin^2\left(\frac{\phi}{2}\right)\cdot(d\phi)^2$$

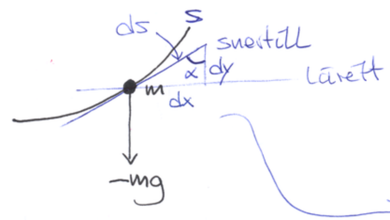
$$\to ds = 2a\sin\left(\frac{\phi}{2}\right)\cdot d\phi$$

(6)

$$\to v = \frac{ds}{dt} = 2a\sin\left(\frac{\phi}{2}\right)\frac{d\phi}{dt} = -4a\frac{d}{dt}\left\{\cos\left(\frac{\phi}{2}\right)\right\}$$

$$\to \dot{v} = -4a\frac{d^2}{dt^2}\left\{\cos\left(\frac{\phi}{2}\right)\right\}$$

Hvað með Kraftinn



$$F = -mg\cos\alpha$$

þurfum að tengja α og ϕ

$$\to \frac{dy}{ds} = \cos\alpha \qquad (**)$$

En $(*)$

$$\frac{dy}{ds} = -\frac{a\sin\phi\cdot d\phi}{2a\sin\left(\frac{\phi}{2}\right)d\phi} = -\frac{\sin\phi}{2\sin\left(\frac{\phi}{2}\right)}$$

(7)

Hálfa hornið þurftar aðeins

en

$$\left(2\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\right)^2 = (1-\cos\phi)(1+\cos\phi) = 1-\cos^2\phi = \sin^2\phi.$$

$$\to -\frac{\sin\phi}{2\sin\left(\frac{\phi}{2}\right)} = -\cos\left(\frac{\phi}{2}\right) \quad \to \quad \frac{dy}{ds} = -\cos\left(\frac{\phi}{2}\right)$$

$$= \cos\alpha \qquad \text{Notum } (**)$$

því er

$$F = +mg\cos\left(\frac{\phi}{2}\right), \qquad m\dot{v} = -4am\frac{d^2}{dt^2}\left\{\cos\left(\frac{\phi}{2}\right)\right\}$$

Skiptum um breytu $z = \cos\left(\frac{\phi}{2}\right)$

$$\to -4a\ddot{z} = gz \quad \to \quad \boxed{\ddot{z} + \left\{\frac{g}{4a}\right\}z = 0}$$

$\to$ hreintóna sveifill með $\omega_0 = \sqrt{\frac{g}{4a}}$ óháð útslagi

(8)

(4) Dæmi 3-11 í bók, deyfður hreintóna sveifill

(sambærilegt við fyrirlestur 3 bls. 16)

$$E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \qquad \omega_1 = \sqrt{\omega_0^2 + \beta^2}$$

$$x = Ae^{-\beta t}\cos(\omega_1 t - \delta) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\dot{x} = Ae^{-\beta t}\left\{-\beta\cos(\omega_1 t - \delta) - \omega_1\sin(\omega_1 t - \delta)\right\}$$

$$\to E(t) = \frac{A^2}{2}e^{-2\beta t}\Big\{(m\beta^2 + k)\cos^2(\omega_1 t - \delta) + m\omega_1^2\sin^2(\omega_1 t - \delta)$$

$$+ 2m\beta\omega_1\sin(\omega_1 t - \delta)\cos(\omega_1 t - \delta)\Big\}$$

Notum

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

og

$$\to E(t) = \frac{mA^2}{2}e^{-2\beta t}\left\{\beta^2\cos(2(\omega_1 t - \delta)) + \beta\sqrt{\omega_0^2 - \beta^2}\sin(2(\omega_1 t - \delta)) + \omega_0^2\right\}$$

$$\rightarrow \dot{E}(t) = \frac{mA^2}{2} e^{-2\beta t}\Big\{ -2\omega_1\beta^2 \sin(2(\omega_1 t-\delta)) + 2\omega_1\beta\sqrt{\omega_0^2-\beta^2}\cos(2(\omega_1 t-\delta))$$
$$- 2\beta\beta^2\cos(2(\omega_1 t-\delta)) - 2\beta^2\sqrt{\omega_0^2-\beta^2}\sin(2(\omega_1 t-\delta))$$
$$- 2\beta\omega_0^2 \Big\}$$

$$= \frac{mA^2}{2} e^{-2\beta t}\Big\{ (2\beta\omega_0^2 - 4\beta^3)\cos(2(\omega_1 t-\delta)) - 4\beta\sqrt{\omega_0^2-\beta^2}\sin(2(\omega_1 t-\delta)) - 2\beta\omega_0^2 \Big\}$$

$$\langle \dot{E}(t)\rangle_{lota} = -\frac{mA^2}{2} 2\beta\omega_0^2 e^{-2\beta t} \quad \text{ef } \beta \ll \omega_0$$
$$= -m\beta\omega_0^2 A^2 e^{-2\beta t}$$

---

6) Dæmi 3-4 í bók

Hreintóna sveifill

$$x = A\sin(\omega_0 t) \qquad \text{með } \omega_0 = \sqrt{\frac{k}{m}}$$
$$\dot{x} = \omega_0 A\cos(\omega_0 t)$$

Lotan er $\tau = \frac{2\pi}{\omega_0}$ → tíma meðaltal yfir eina lotu

$$\langle T\rangle_t = \frac{1}{\tau}\int_t^{t+\tau} dt' \frac{m}{2}\dot{x}^2 = \frac{m}{2\tau}A^2\omega_0^2 \int_t^{t+\tau}\cos^2(\omega_0 t')\,dt' \quad \left(= \frac{\tau}{2}\right)$$
$$= \frac{mA^2\omega_0^2}{4}$$

$$\langle U\rangle_t = \frac{1}{\tau}\int_t^{t+\tau} dt' \frac{kx^2}{2} = \frac{k}{2\tau}A^2\int_t^{t+\tau}\sin^2(\omega_0 t')\,dt' = \frac{kA^2}{4} = \frac{mA^2\omega_0^2}{4}$$

$$\rightarrow \langle T\rangle_t = \langle U\rangle_t$$

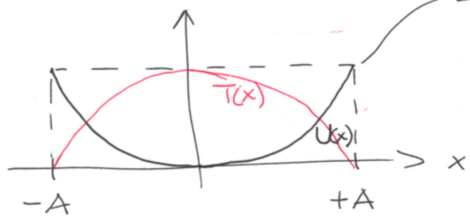
$$\langle T\rangle_x = \frac{1}{A}\int_0^A \frac{m}{2}\dot{x}^2\,dx \;, \quad \langle U\rangle_x = \frac{1}{A}\int_0^A \frac{kx^2}{2}\,dx$$

$$\langle U\rangle_x = \frac{m\omega_0^2 A^2}{6}$$ og síðan notum

$$\dot{x}^2 = \omega_0^2 A^2\cos^2(\omega_0 t)$$
$$= \omega_0^2 A^2\{1 - \sin^2(\omega_0 t)\}$$
$$= \omega_0^2 (A^2 - x^2)$$

$$\rightarrow \langle T\rangle_x = 2\frac{m\omega_0^2 A^2}{6} = 2\langle U\rangle_x$$

Af hverju

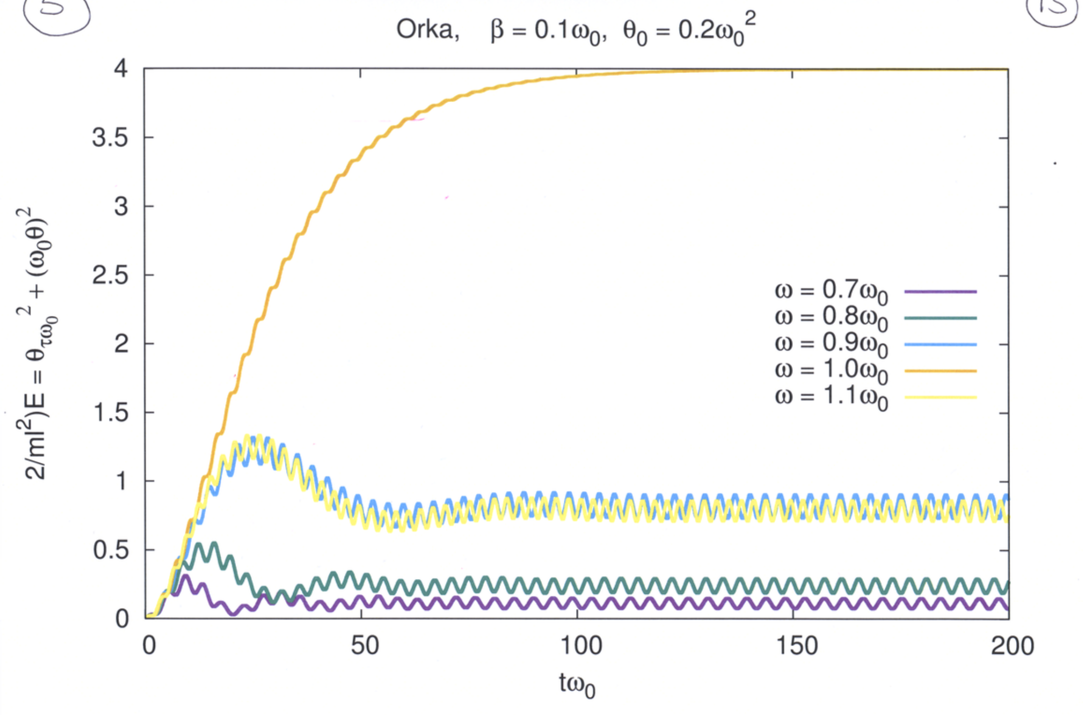


$E_{heild} = \frac{1}{2} m A^2 \omega_0^2 = \text{fasti}$

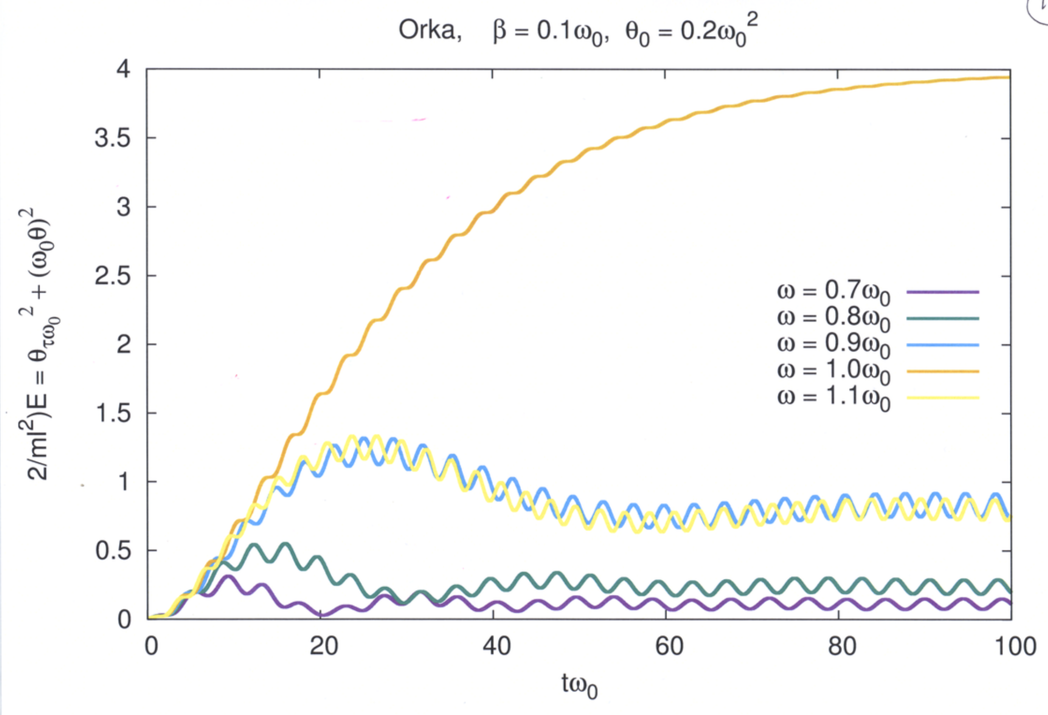
Flatarmálið er mism

5

13


Orka, β = 0.1ω0, θ0 = 0.2ω0^2
2/ml^2)E = θτω0^2 + (ω0θ)^2
tω0
ω = 0.7ω0
ω = 0.8ω0
ω = 0.9ω0
ω = 1.0ω0
ω = 1.1ω0
0
0.5
1
1.5
2
2.5
3
3.5
4
0
50
100
150
200

14


Orka, β = 0.1ω0, θ0 = 0.2ω0^2
2/ml^2)E = θτω0^2 + (ω0θ)^2
tω0
ω = 0.7ω0
ω = 0.8ω0
ω = 0.9ω0
ω = 1.0ω0
ω = 1.1ω0
0
0.5
1
1.5
2
2.5
3
3.5
4
0
20
40
60
80
100

(1) Hreyfijafna

$$m\dot{\bar v} = \bar F_{ext} + \bar F_{rad}\,, \qquad \bar F_{rad} = m\tau\ddot{\bar v}$$

-> hreyfijafna kerfisins er

$$m\left\{\dot{\bar v} - \tau\ddot{\bar v}\right\} = \bar F_{ext}$$

þegar $\bar F_{ext} = 0$ fæst

$$\dot{\bar v} - \tau\ddot{\bar v} = 0 \quad \to \quad \ddot{\bar v} = \frac{1}{\tau}\dot{\bar v}$$

Umskrifum

$$\dot{\bar a} = \frac{1}{\tau}\bar a \quad \to \quad \frac{d\bar a}{dt} = \frac{1}{\tau}\bar a$$

$$\to \quad \frac{da}{a} = \frac{1}{\tau}dt \quad \to \quad \int_{a_0}^{a}\frac{da'}{a'} = \frac{1}{\tau}\int_0^t dt'$$

(2)

$$\ln\left[\frac{a}{a_0}\right] = \frac{1}{\tau}t \quad \to \quad \bar a(t) = \bar a_0 e^{t/\tau}$$

Ef $\bar a_0 \neq 0$ þá er þetta „run away solution" sem er ekki eðlisfræðileg. Því verðum við að segja að $\bar F_{ext} = 0$ gefi okkur að eindinni sé ekki hraðað -> engin geislun.

Abraham-Lorentz jafnan er óvenjuleg. Fræðileg lausn á henni leiðir til forhröðunar sem verður á tímanum $-\tau \to 0$ sem uppfyllir ekki orsakar lögmálið. Fyrir rafeindir $\tau \sim 10^{-23}$ s þar búumst við ekki við góðri klassískri lýsingu.....

(3)

Dæmi 4-4 í bók

Lord Rayleigh

$$\ddot x - (a - b\dot x^2)\dot x + \omega_0^2 x = 0$$

Athugum breytuskiptin $y = y_0\sqrt{\frac{3b}{a}}\,\dot x$ eftir tímaafleiðu

Tímaafleiða

$$\dddot x - (a - b\dot x^2)\ddot x + 2b\dot x\ddot x\dot x + \omega_0^2\dot x = 0$$

$$\to \quad \dddot x - (a - 3b\dot x^2)\ddot x + \omega_0^2\dot x = 0$$

Innsetn.

$$y = y_0\sqrt{\frac{3b}{a}}\,\dot x \quad \to \quad \dot y = y_0\sqrt{\frac{3b}{a}}\,\ddot x\,, \quad \ddot y = y_0\sqrt{\frac{3b}{a}}\,\dddot x$$

$$\to \quad \dot x = \sqrt{\frac{a}{3b}}\,\frac{y}{y_0}\,, \quad \ddot x = \sqrt{\frac{a}{3b}}\,\frac{\dot y}{y_0}\,, \quad \dddot x = \sqrt{\frac{a}{3b}}\,\frac{\ddot y}{y_0}$$

(4)

Setjum inn í jöfnu

$$\frac{1}{y_0}\sqrt{\frac{a}{3b}}\left\{\ddot y - \left[a - a\frac{y^2}{y_0^2}\right]\dot y + \omega_0^2 y\right\} = 0$$

sem verður

$$\ddot y - a\left(1 - \frac{y^2}{y_0^2}\right)\dot y + \omega_0^2 y = 0$$

$$\ddot y + \frac{a}{y_0^2}\left(y^2 - y_0^2\right)\dot y + \omega_0^2 y = 0$$

Jafna van der Pols var

$$\ddot x + \mu\left(x^2 - a'^2\right)\dot x + \omega_0^2 x = 0$$

-> í okkar jöfnu er

$$\frac{a}{y_0^2} = \mu \quad \text{og} \quad y_0^2 = a'$$

③ Jafna van der Pols ⑤

$$\ddot{x} + \mu(x^2-a^2)\dot{x} + \omega_0^2 x = 0$$

setjum sem svo að x og a sēu víddarlausar stærðir
μ getur ekki verið það

$$[\omega_0] = \frac{1}{T} \quad \text{og} \quad [\mu] = \frac{1}{T}$$

því er aðeins hægt að ræða hvað gerist þegar $\frac{\mu}{\omega_0}$ er smá stærð, endurritum

$$\ddot{x} + \omega_0^2 x = -\frac{\mu}{\omega_0}(x^2-a^2)\omega_0\dot{x} = \frac{\mu}{\omega_0}(a^2-x^2)\omega_0\dot{x}$$

Ótruflaða lausnin þegar $\mu = 0$ gæti verið

$$x(t) = b\cos(\omega_0 t)$$

hreintóna sveifill með tíðni ω_0

⑥

Gerum ráð fyrir að lausnin fyrir smátt μ/ω_0 sé

$$x(t) = b\cos(\omega_0 t) + u(t)$$

Athugum jöfnu fyrir u:

$$0 + \ddot{u} + \omega_0^2 u = \frac{\mu}{\omega_0}\left\{a^2 - \left[b\cos(\omega_0 t) + u\right]^2\right\}\omega_0\left\{-b\omega_0\sin(\omega_0 t) + \dot{u}\right\}$$

$$\to \ddot{u} + \omega_0^2 u = \frac{\mu}{\omega_0}\left\{a^2 - b^2\cos^2(\omega_0 t) - 2bu\cos(\omega_0 t) - u^2\right\} \cdot \omega_0\left\{-b\omega_0\sin(\omega_0 t) + \dot{u}\right\}$$

u er í réttu hlutfalli við $\frac{\mu}{\omega_0}$, sleppum því liðum með u-i hægramegin
← aðminnsta kosti

⑦

$$\ddot{u} + \omega_0^2 u = -\frac{\mu}{\omega_0}\left\{\omega_0\left(a^2 - b^2\cos^2(\omega_0 t)\right) b\omega_0\sin(\omega_0 t)\right\}$$

$$= -\frac{\mu}{\omega_0}\left\{\omega_0^2\left[a^2 b\sin(\omega_0 t) - b^3\cos^2(\omega_0 t)\sin(\omega_0 t)\right]\right\}$$

notum

$$\cos^2(\omega_0 t)\sin(\omega_0 t) = \frac{1}{3}\left\{\sin(3\omega_0 t) + \sin^3(\omega_0 t)\right\}$$

$$= \frac{1}{3}\left\{\sin(3\omega_0 t) + \frac{3\sin(\omega_0 t) - \sin(3\omega_0 t)}{4}\right\}$$

$$= \frac{1}{4}\sin(3\omega_0 t) + \frac{1}{4}\sin(\omega_0 t)$$

því fæst

$$\ddot{u} + \omega_0^2 u = -\frac{\mu}{\omega_0}\omega_0^2 b\left\{a^2\sin(\omega_0 t) - \frac{b^2}{4}\sin(\omega_0 t) - \frac{b^2}{4}\sin(3\omega_0 t)\right\}$$

$$= -\frac{\mu}{\omega_0}\omega_0^2 b\left\{\left(a^2 - \frac{b^2}{4}\right)\sin(\omega_0 t) - \frac{b^2}{4}\sin(3\omega_0 t)\right\}$$

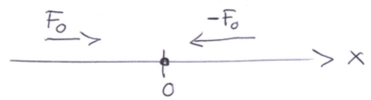
⑧

því sést að í línulegri nálgun fyrir $\frac{\mu}{\omega_0}$ er jafnan eins og jafna hreintóna sveifils með tíðni ω_0, en í henni eru þvingunarliðir með tíðni ω_0 og $3\omega_0$

④ dæmi 4-8 í bók

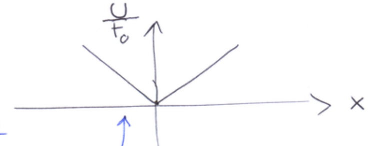
Engin deyfing

1. D-hreyfing



$$F = -\frac{\partial}{\partial x}U(x) \qquad F = \begin{cases} F_0 & \text{ef } x < 0 \\ -F_0 & \text{ef } x > 0 \end{cases}$$

$$\to U(x) = \begin{cases} -F_0 x & \text{ef } x < 0 \\ +F_0 x & \text{ef } x > 0 \end{cases} \qquad \to U(x) = F_0|x|$$



Hver grein gæti verið mættið í föstu þyngdarsviði

(9)

Veljum $x(0) = A$, $\dot{x}(0) = 0$, mesta útslag
Hreyfijafnan er ekki heppileg vegna brotsins, en

$$E = \frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv^2 + F_0|x| \quad , \quad |x| \leq A$$

upphafsgildin gefa að $\quad E = F_0 A$

$$\rightarrow \frac{1}{2}mv^2 + F_0|x| = F_0 A \rightarrow v^2 = \frac{2F_0}{m}(A-|x|)$$

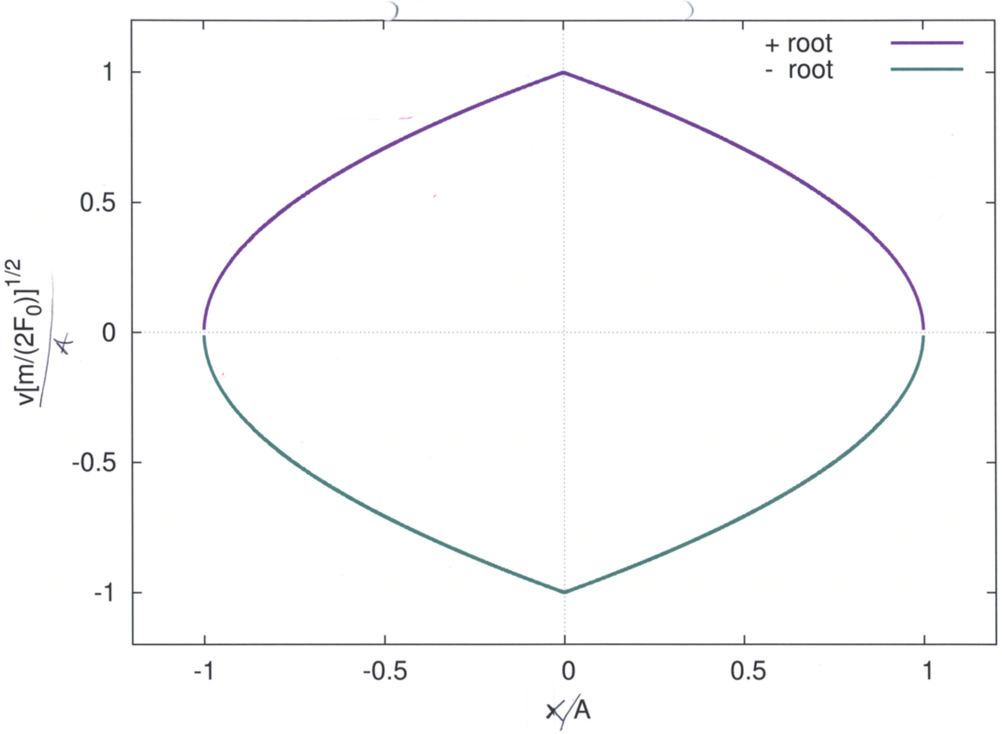
$$\rightarrow v = \pm\sqrt{\frac{2F_0}{m}(A-|x|)}$$

fyrir graf

$$v = \pm A\sqrt{\frac{2F_0}{m}}\sqrt{\left(1-\frac{|x|}{A}\right)} \rightarrow \frac{\dot{x}(x)}{A}\cdot\sqrt{\frac{m}{2F_0}} = \sqrt{1-\frac{|x|}{A}}$$

Sjá næstu síðu, þar sem fasarit er sýnt

(10)



(11)

Finnum lotuna

Sveiflan er samhverf
Skoðum hreyfijöfnuna fyrir $x > 0$

$$m\ddot{x} = -F_0$$

finnum $\tau/4$ þegar sveifillinn fer frá $x = A$ í $x = 0$

$$\ddot{x} = \frac{d^2x}{dt^2} = -\frac{F_0}{m} = \frac{dv}{dt}$$

$$\rightarrow \int_0^v dv' = -\frac{F_0}{m}\int_0^t dt' \rightarrow v(t) = -\frac{F_0}{m}t = \frac{dx}{dt}$$

$$\rightarrow \int_A^0 dx = -\frac{F_0}{m}\int_0^{\tau/4} t'dt' \rightarrow A = \frac{F_0\tau^2}{2m16} \rightarrow \tau^2 = \frac{32mA}{F_0}$$

$$\rightarrow \tau = 4\cdot\sqrt{\frac{2mA}{F_0}}$$

(12)

(5) Þvingaði deyfði sveifillinn ~~eftir~~ svipulausnina um hann gildir

$$\dot{x} = \frac{-A\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\sin(\omega t-\delta)$$

{samanber jöfnu (3.60) í bók}

því er „hraða útslagið"

$$v_{max} = \frac{A\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

Við viljum vita fyrir hvaða $\omega = \omega_v$ fæst max gildi

$$\left.\frac{\partial v_{max}}{\partial\omega}\right|_{\omega=\omega_v} = 0$$

(13)

$$\frac{\partial U_{max}}{\partial \omega} = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}} - \frac{\frac{1}{2}A\omega\left\{4\omega^3-4\omega_0^2\omega+8\beta^2\omega\right\}}{\left[(\omega_0^2-\omega^2)^2+4\omega^2\beta^2\right]^{3/2}} = 0$$

$$\rightarrow \left\{\omega_0^4+\omega^4-2\omega_0^2\omega^2+4\omega^2\beta^2-2\omega^4+2\omega_0^2\omega^2-4\beta^2\omega^2\right\} = 0$$

$$\rightarrow \omega_0^4-\omega^4 = 0 \qquad \rightarrow \omega = \omega_0$$

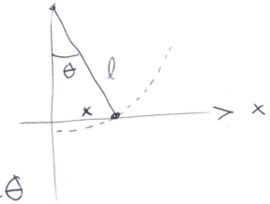
Þetta öllu heldur $\omega_v = \omega_0$ er hermutíðni hraða

Hermutíðnin fyrir útslagið var $\omega_R^2 = \omega_0^2 - 2\beta^2$

(14)

(6) Dæmi 4-7 í bók

pendúll $\ddot{\theta} + \omega_0^2 \sin\theta = 0$ $\quad \omega_0^2 = \frac{g}{l}$



Skoðum lárétta hreyfingu $\quad x = l\sin\theta$

$$\sin\theta = \frac{x}{l} \rightarrow \theta = \arcsin\left(\frac{x}{l}\right)$$

$$\rightarrow \dot{\theta} = \frac{1}{\sqrt{1-\left(\frac{x}{l}\right)^2}}\frac{\dot{x}}{l} \qquad \rightarrow \ddot{\theta} = \frac{\ddot{x}}{l}\frac{1}{\sqrt{1-\left(\frac{x}{l}\right)^2}} + \frac{\frac{x}{l}\frac{\dot{x}}{l}\frac{\dot{x}}{l}}{\left(1-\left(\frac{x}{l}\right)^2\right)^{3/2}}$$

$\left(\frac{\dot{x}}{l}\right)^2$ minnir á orku $\qquad$ heildarorka $\qquad$ max stöðuorka

$$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = E = -mgl\cos\theta_0$$

$$\frac{1}{2}\dot{\theta}^2 - \omega_0^2\cos\theta = -\omega_0^2\cos\theta_0$$

$$\rightarrow \dot{\theta}^2 = 2\omega_0^2\left\{\cos\theta - \cos\theta_0\right\}$$

(15)

endurritum sem

$$\frac{1}{1-\left(\frac{x}{l}\right)^2}\left(\frac{\dot{x}}{l}\right)^2 = 2\omega_0^2\left\{\sqrt{1-\left(\frac{x}{l}\right)^2} - \sqrt{1-\left(\frac{x_0}{l}\right)^2}\right\}$$

Setjum saman

$$\ddot{\theta} + \omega_0^2\sin\theta = 0$$

$$\frac{\ddot{x}}{l}\frac{1}{\sqrt{1-\left(\frac{x}{l}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x}{l}\right)^2}}\frac{x}{l}\left\{\sqrt{1-\left(\frac{x}{l}\right)^2} - \sqrt{1-\left(\frac{x_0}{l}\right)^2}\right\}2\omega_0^2 + \omega_0^2\frac{x}{l} = 0$$

Viljum halda liðum upp í $(x/l)^3$

$$\frac{\ddot{x}}{l} + \frac{x}{l}\left\{\sqrt{1-\left(\frac{x}{l}\right)^2} - \sqrt{1-\left(\frac{x_0}{l}\right)^2}\right\}2\omega_0^2 + \omega_0^2\frac{x}{l}\sqrt{1-\left(\frac{x}{l}\right)^2} = 0$$

Notum $\sqrt{1-\left(\frac{x}{l}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{1}{8}\left(\frac{x}{l}\right)^4 + \cdots\cdots$

(16)

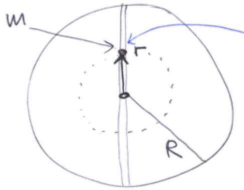
$$\frac{\ddot{x}}{l} + \frac{x}{l}\left\{-\frac{1}{2}\left(\frac{x}{l}\right)^2 + \frac{1}{2}\left(\frac{x_0}{l}\right)^2\right\}2\omega_0^2 + \omega_0^2\frac{x}{l}\left\{1 - \frac{1}{2}\left(\frac{x}{l}\right)^2\right\} = 0$$

$$\rightarrow \boxed{\frac{\ddot{x}}{l} + \omega_0^2\left\{1 + \left(\frac{x_0}{l}\right)^2\right\}\frac{x}{l} - \frac{3}{2}\omega_0^2\left(\frac{x}{l}\right)^3 = 0}$$

Í bókinni er skrifað eftir margföldun með l

$$\ddot{x} + \omega_0^2\left\{1 - \left(\frac{x_0}{l}\right)^2\right\}x - \frac{3}{2}\frac{g}{l^3}x^3 = 0$$

① Dæmi 5-15 í bók.



ρ er fasti, þéttleiki jarðar í þessu líkani

Lögmál Gauß → aðeins massinn innan geislans r, $M(r)$, veldur krafti á ögnina að miðju

$$F(r) = -\frac{GmM(r)}{r^2}, \quad M(r) = \frac{4}{3}\pi r^3 \rho$$

$$= -Gmr\frac{4\pi\rho}{3} = -G\frac{m4\pi\rho}{3} r = m\ddot{r}$$

$$\to \ddot{r} + G\frac{4\pi\rho}{3} r = 0 \quad \text{eða} \quad \boxed{\ddot{r} + \omega_0^2 r = 0}$$

Hreintóna sveifla með $\omega_0 = \sqrt{\frac{4\pi G\rho}{3}} = 2\pi\sqrt{\frac{G\rho}{3\pi}}$

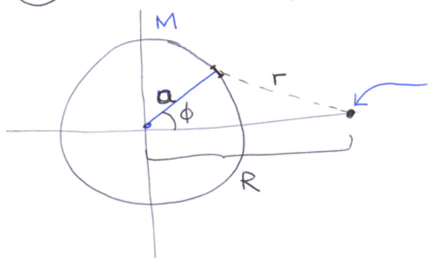
Lota $\boxed{\tau = \frac{2\pi}{\omega_0} = \sqrt{\frac{3\pi}{G\rho}}}$

$\rho = 5.514\ \mathrm{g/cm}$, $G = 6{,}674\cdot 10^{-8}\ \frac{\mathrm{cm}^3}{\mathrm{g\,s}^2}$

$$\to \tau = \sqrt{\frac{3\pi}{5.514\cdot 6{,}674\cdot 10^{-8}}}\ \mathrm{s} \simeq 5061\ \mathrm{s} \approx 84\ \text{mín}$$

Ef $\rho = \rho(r)$ má finna $M(r) = 4\pi\int_0^r x^2 dx\, \rho(x)$

② dæmi 5-9 í bók. Reikna mættið Φ fyrir punkt í sléttu hringsins



$$d\Phi = -G\frac{dM}{r} = -G\frac{\rho_\ell d\ell}{r}$$

$$d\ell = a\,d\phi$$

$$r = \sqrt{R^2 + a^2 - 2aR\cos\phi}$$

$$\rho_\ell = \frac{M}{2\pi a}$$

$$\Phi(R) = -\frac{GM}{2\pi}\int_0^{2\pi}\frac{d\phi}{\sqrt{R^2 + a^2 - 2aR\cos\phi}} = -\frac{GM}{2\pi R}\int_0^{2\pi}\frac{d\phi}{\sqrt{1 + \left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\cos\phi}}$$

Athugum liðinn fyrir $\left(\frac{a}{R}\right) \ll 1$, notum $\frac{1}{\sqrt{1+x}} \simeq 1 - \frac{x}{2} + \frac{3x^2}{8} + \dots$

$$\to \Phi(R) \simeq -\frac{GM}{2\pi R}\int_0^{2\pi} d\phi\left[1 - \frac{1}{2}\left\{\left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\cos\phi\right\} + \frac{3}{8}\left\{\left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\cos\phi\right\}^2 + \dots\right]$$

höldum $\left(\frac{a}{R}\right)^2$-liðum

$$\to \boxed{\Phi(R) \simeq} -\frac{GM}{2\pi R}\int_0^{2\pi} d\phi\left\{1 - \frac{1}{2}\left(\frac{a}{R}\right)^2 + \underbrace{\left(\frac{a}{R}\right)\cos\phi}_{\to 0} + \frac{3}{2}\left(\frac{a}{R}\right)^2\cos^2\phi\right\}$$

$$= -\frac{GM}{2\pi R}\left\{2\pi - \pi\left(\frac{a}{R}\right)^2 + \frac{3}{2}\left(\frac{a}{R}\right)^2\pi\right\}$$

$$= \boxed{-\frac{GM}{R}\left\{1 + \frac{1}{4}\left(\frac{a}{R}\right)^2\right\}} \quad \text{ef} \quad \left(\frac{a}{R}\right) \ll 1$$

③ Kúla með massadreifingu $\rho(r)$

Við vitum að $\bar{g} = -\nabla\Phi$ og $\nabla^2\Phi = 4\pi G\rho(r)$

Ef ρ er aðeins háð r → Φ er aðeins fall af r

Í kúluhnitum jafna Poissons

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 4\pi G\rho(r)$$

eins er $\bar{g} = -\nabla\Phi = -\hat{e}_r\frac{\partial}{\partial r}\Phi(r)$

Ef $\bar{g}$ er óháð r, þá er $\frac{\partial}{\partial r}\Phi(r) = -g_0 = \text{fasti}$

Ef Φ er aðeins háð r, þá er g ekki háð ϕ eða θ

(5)

$\frac{\partial}{\partial r}\Phi(r) = g_0$, notum í jöfnu Poissons

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g_0\right) = 4\pi G\rho(r)$$

$$-g_0 \frac{2}{r^2} r = -g_0 \frac{2}{r} \quad \rightarrow \quad -g_0 \frac{2}{r} = 4\pi G\rho(r)$$

$$\rightarrow \rho(r) = -\frac{g_0}{2\pi G r}$$

Við búumst því við að $g_0 < 0$

Í miðju kúlunnar er sérstöðupunktur fyrir massann, af sömu tegund og punktmassi hefði, ef ekki þá hefði g þurft að vera háð r þ.a. $\bar{g}(r) \to 0$ þegar $r \to 0$

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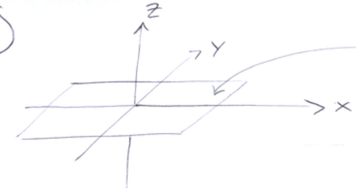
(6)

(4)



massaþéttleiki ρ_s

$[\rho_s] = \frac{M}{L^2}$ ← massi á flöt

Finna þyngdarmættið Φ og sviðið $\bar{g}$

$$\nabla^2\Phi = 4\pi G\rho, \quad \bar{g} = -\nabla\Phi$$

Notum lögmál Gauß

$$\oint_S \bar{n}\cdot\bar{g}\, da = -4\pi G\int_V \rho\, dv = -4\pi G M_{encl}$$

það er aðdráttur að sléttunni, samhverfa hægra og vinstra megin, vegna samhverfu getur $\bar{g}$ aðeins verið hornrétt á sléttuna.

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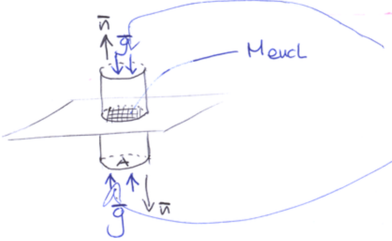
(7)

Hugsum okkur lokað Gauß yfirborð sem sívalning þvert á sléttuna



Eina flæðið er um topp og botn sívalningsins, ekkert um bogna flötinn! Flæðið er óháð hæð sívalnings

$$\oint_S \bar{n}\cdot\bar{g}\, da = -2Ag$$

$$-4\pi G M_{encl} = -4\pi G(\rho_s A)$$

$$\rightarrow g = 2\pi G\rho_s$$

því fæst $\bar{g} = -\hat{e}_z \frac{z}{|z|} 2\pi G\rho_s$

fast þyngdarsvið óháð fjarlægð frá sléttu sitthvoru megin með stefnu að sléttu

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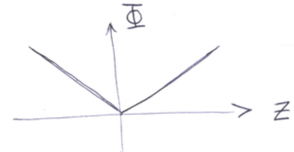
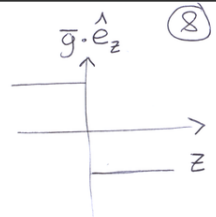
(8)

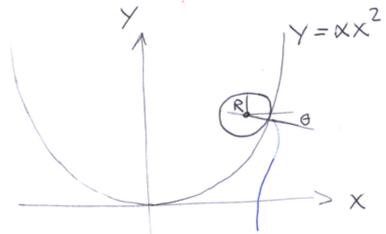
$$\bar{g} = -\nabla\Phi \quad \rightarrow \quad \Phi = |z|\, 2\pi G\rho_s$$



brot í Φ og þrep í $\bar{g}$ vegna massasléttu



(5) Dæmi 6-11 í bók



Hvernig eru skilyrðin fyrir því að skífan velti þ.a. hún snerti fleygbogann aðeins í einum punkti

$$ds = R\,d\theta$$

$$\hookrightarrow = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$y = \alpha x^2 \;\rightarrow\; \frac{dy}{dx} = 2\alpha x$$

$$\rightarrow ds = \sqrt{1+(2\alpha x)^2}\, dx = R\, d\theta \;,\; \text{heildum}$$

$$R\int d\theta = \int \sqrt{1+(2\alpha x)^2}\, dx \quad \leftarrow \text{stærður } \theta(x)$$

$$C + R\theta = \frac{x\sqrt{1+(2\alpha x)^2}}{2} + \frac{\mathrm{ArSinh}(2\alpha x)}{4\alpha} \qquad \alpha > 0$$

↑ heildunarfasti ákvarðast af upphafsskilyrðum

Fyrir fall $y = f(x)$ er sveigjugeisli (radius of curvature)

$$\frac{1}{r_0} = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

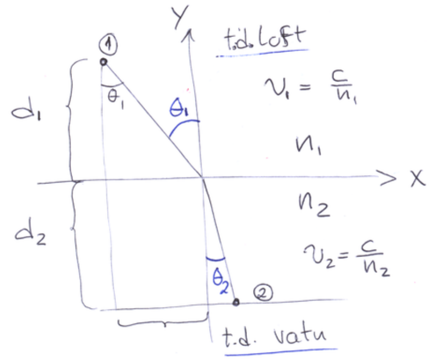
$\rightarrow r_0 \geq \frac{1}{2\alpha}$ fyrir alla punkta (x,y) á fleygboganum

$$= \frac{2\alpha}{(1+(2\alpha x)^2)^{3/2}} \quad \text{fyrir fleygbogann}$$

→ skífan veltur alls staðar ef $R < r_0 \geq \frac{1}{2\alpha}$

$$\rightarrow \boxed{R < \frac{1}{2\alpha}}$$

(6) Dæmi 6-7 í bók



$n_2 > n_1$

Lágmarka tíma til þess að leiða út $n_1 \sin\theta_1 = n_2 \sin\theta_2$

$$\frac{ds}{dt} = v \;\rightarrow\; dt = \frac{ds}{v}$$

$$\int_1^2 dt = \int_1^2 \frac{ds}{v}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\rightarrow t_2 - t_1 = \Delta t = \int_1^2 \frac{\sqrt{1+(y')^2}}{v}\, dx$$

Notum Euler $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \;,\; f(y,y') = \frac{\sqrt{1+(y')^2}}{v}$

$v = v(y)$, en $\frac{dv}{dy} = 0$ nema í punktinum $y = 0$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{v\sqrt{1+(y')^2}} \;\rightarrow\; \frac{d}{dx}\left\{\frac{y'}{v\sqrt{1+(y')^2}}\right\} = 0$$

Vitum að $v_i = \frac{c}{n_i}$

$$\frac{dy_i}{dx} = -\tan\theta_i = y'$$

$$\frac{-\tan\theta_i \cdot n_i}{c\sqrt{1+\tan^2\theta_i}} = C_1 \leftarrow \text{fasti}$$

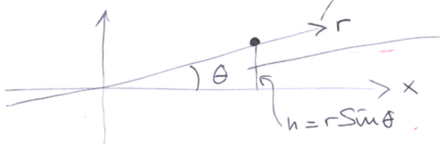
$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

$$\frac{-\tan\theta_i \cdot n_i}{c\sqrt{1+\tan^2\theta_i}} = C_1 \;\rightarrow\; -\frac{n_i}{c}\sin\theta_i = C_1 \;\text{ fasti}$$

Lögmál Snells

① 7-12 í bók ①

tvívíð sletta, hnit ϕ og r



$\theta(t) = \alpha t$

$\bar{v}(0) = 0$ fyrir ögn

Finna hreyfingu agnar. Ef hnitakerfinu er snúið svona verður engin hreyfing í ϕ-stefnu, enginn kraftar.

það eina sem breytist er $U(r,\theta,t) = mgh = mgr\,\mathrm{Sin}\,\theta(t) = mgr\,\mathrm{Sin}(\alpha t)$

$$T = \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} = \frac{m}{2}\left\{\dot{r}^2 + (r\alpha)^2\right\}$$

$$\to L = \frac{m}{2}\left\{\dot{r}^2 + (r\alpha)^2\right\} - mgr\,\mathrm{Sin}(\alpha t)$$

eitt alhnit r

②

notum Lagrange

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \to mr\alpha^2 - mg\,\mathrm{Sin}(\alpha t) - m\ddot{r} = 0$$

Hreyfijafnan er því

$$\ddot{r} - r\alpha^2 + g\,\mathrm{Sin}(\alpha t) = 0 \quad \text{eða} \quad \boxed{\ddot{r} - \alpha^2 r = -g\,\mathrm{Sin}(\alpha t)}$$

Almenna lausn óhliðruðu jöfnunnar er

$$r_h(t) = Ae^{\alpha t} + Be^{-\alpha t}$$

Giskum á sérlausn $r_p(t) = C\,\mathrm{Sin}(\alpha t)$

reynum í hreyfijöfnu

$$-C\alpha^2\,\mathrm{Sin}(\alpha t) - C\alpha^2\,\mathrm{Sin}(\alpha t) = -g\,\mathrm{Sin}(\alpha t)$$

$$\to 2C\alpha^2 = g \quad \text{eða} \quad C = \frac{g}{2\alpha^2}$$

③

Heildarlausnin er því

$$\boxed{r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2}\mathrm{Sin}(\alpha t)}$$

með jaðarskilyrðum

$$\dot{r}(0) = 0 \qquad r(0) = r_0$$

$$0 = A\alpha - B\alpha + \frac{g}{2\alpha}$$

$$r_0 = A + B$$

eða

$$A - B = -\frac{g}{2\alpha^2}$$
$$A + B = r_0$$

$$\to \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{g}{2\alpha^2} \\ r_0 \end{pmatrix}$$

$$\to \quad A = -\frac{1}{2}\left\{\frac{g}{2\alpha^2} - r_0\right\} = \frac{1}{2}\left\{r_0 - \frac{g}{2\alpha^2}\right\}$$

$$B = \frac{1}{2}\left\{\frac{g}{2\alpha^2} + r_0\right\} = \frac{1}{2}\left\{r_0 + \frac{g}{2\alpha^2}\right\}$$

④

$$\to r(t) = \frac{1}{2}\left[\left\{r_0 - \frac{g}{2\alpha^2}\right\}e^{\alpha t} + \left\{r_0 + \frac{g}{2\alpha^2}\right\}e^{-\alpha t} + \frac{g}{\alpha^2}\mathrm{Sin}(\alpha t)\right]$$

$$= r_0\,\mathrm{Cosh}(\alpha t) + \frac{g}{2\alpha^2}\left\{\mathrm{Sin}(\alpha t) - \mathrm{Sinh}(\alpha t)\right\}$$

Til umhugsunar, ef t.d. $r_0 = 0$ í upphafi hvað gerist þegar $\alpha t > \frac{\pi}{2}$ og þegar $\alpha t \to \infty$

② dæmi 7-15 í bók

b í jafnvægi

Alhnit gætu verið θ og tímaháð lengd pendúls l

$$T = \frac{m}{2}\left\{\dot{l}^2 + (l\dot{\theta})^2\right\}$$

$$U = mgz + \frac{1}{2}k(l-b)^2 = -mgl\,\mathrm{Cos}\theta + \frac{1}{2}k(l-b)^2$$

Lagrange jöfnurnar eru því

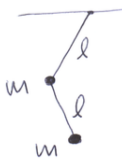
$$\frac{\partial L}{\partial l} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot l}\right) = 0 \;\rightarrow\; ml\dot\theta^2 + mg\cos\theta - k(l-b) - m\ddot l = 0$$

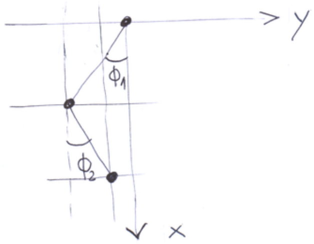
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0 \;\rightarrow\; -mgl\sin\theta - \frac{d}{dt}\left\{2l^2\dot\theta\right\}\frac{m}{2} = 0$$

$$\rightarrow \boxed{\begin{aligned} &\ddot l - l\dot\theta^2 - g\cos\theta + \frac{k}{m}(l-b) = 0 \\ &\ddot\theta + \frac{2\dot l}{l}\dot\theta + \frac{g}{l}\sin\theta = 0 \end{aligned}}$$

③ dæmi 7-07 Tvöfaldur pendúll

sveiflast í sömu sléttu





$$x_1 = l\cos\phi_1 \qquad x_2 = x_1 + l\cos\phi_2$$

$$y_1 = l\sin\phi_1 \qquad y_2 = x_1 + l\sin\phi_2$$

$$T = \frac{m}{2}\left\{\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2\right\}$$

$$= \frac{ml^2}{2}\left\{\dot\phi_1^2 + \dot\phi_1^2 + 2\dot\phi_1\dot\phi_2\left(\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\right) + \dot\phi_2^2\right\}$$

$$= \frac{ml^2}{2}\left\{2\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)\right\}$$

$$U = -mgx_1 - mgx_2$$ ← miðað við okkar hnitakerfi með $U=0$ fyrir $x=0$

$$= -mg\left\{x_1 + x_1 + l\cos\phi_2\right\} = -mgl\left\{2\cos\phi_1 + \cos\phi_2\right\}$$

$$L = T - U = \frac{ml^2}{2}\left\{2\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)\right\} + mgl\left\{2\cos\phi_1 + \cos\phi_2\right\}$$

$$\frac{\partial L}{\partial \phi_1} = -mgl2\sin\phi_1 - \frac{ml^2}{2}2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \dot\phi_1} = 2ml^2\dot\phi_1 + ml^2\dot\phi_2\cos(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \phi_2} = -mgl\sin\phi_2 + ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \dot\phi_2} = ml^2\dot\phi_2 + ml^2\dot\phi_1\cos(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \phi_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_1}\right) = 0 \;\rightarrow\; -mgl2\sin\phi_1 - \cancel{ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)}$$
$$- 2ml^2\ddot\phi_1 - ml^2\ddot\phi_2\cos(\phi_1-\phi_2)$$
$$- ml^2\dot\phi_2^2\sin(\phi_1-\phi_2) + \cancel{ml^2\dot\phi_2\dot\phi_1\sin(\phi_1-\phi_2)} = 0$$

$$-gl2\sin\phi_1 - 2l^2\ddot\phi_1 - l^2\ddot\phi_2\cos(\phi_1-\phi_2) - l^2\dot\phi_2^2\sin(\phi_1-\phi_2) = 0$$

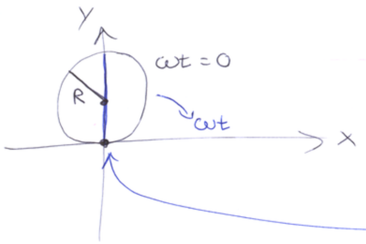
$$\rightarrow \boxed{\ddot\phi_1 + \ddot\phi_2\frac{\cos(\phi_1-\phi_2)}{2} + \dot\phi_2^2\frac{\sin(\phi_1-\phi_2)}{2} + \frac{g}{l}\sin\phi_1 = 0}$$

$$\frac{\partial L}{\partial \phi_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_2}\right) = 0 \;\rightarrow\; -mgl\sin\phi_2 + \cancel{ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)}$$
$$- ml^2\ddot\phi_2 - ml^2\ddot\phi_1\cos(\phi_1-\phi_2)$$
$$+ ml^2\dot\phi_1^2\sin(\phi_1-\phi_2) - \cancel{ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)} = 0$$

$$-gl\sin\phi_2 - l^2\ddot\phi_2 - l^2\ddot\phi_1\cos(\phi_1-\phi_2) + l^2\dot\phi_1^2\sin(\phi_1-\phi_2) = 0$$

$$\rightarrow \boxed{\ddot\phi_2 + \underbrace{\ddot\phi_1\cos(\phi_1-\phi_2)}_{\text{hröðunar víxlverkun}} - \underbrace{\dot\phi_1^2\sin(\phi_1-\phi_2)}_{\text{hraða víxlverkun}} + \frac{g}{l}\sin\phi_2 = 0}$$

(4) Dæmi 7-11 í bók



Veljum stikun

$$x = R\{\cos(\omega t) + \cos(\phi + \omega t)\}$$

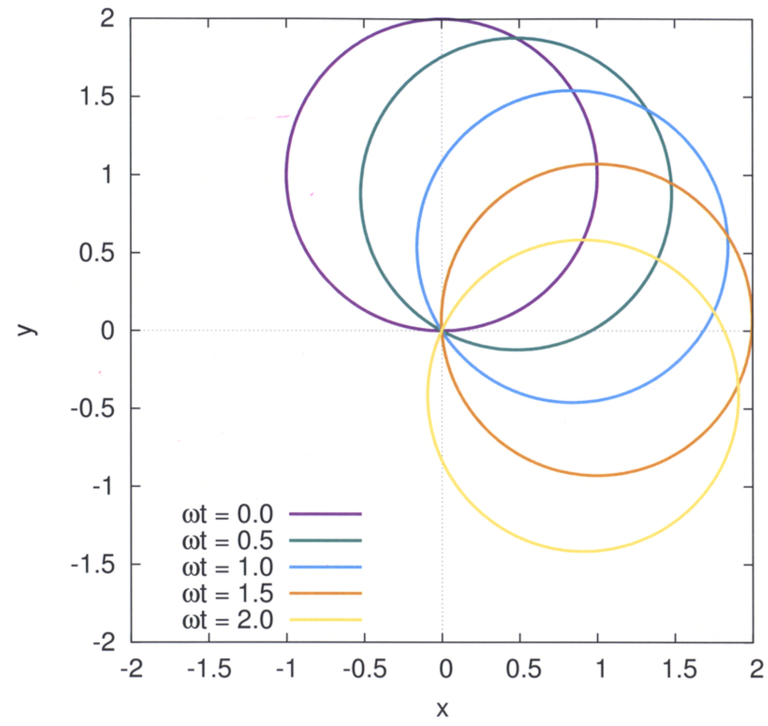
$$y = R\{\sin(\omega t) + \sin(\phi + \omega t)\}$$

Ögn á hring sem snýst um punkt á hring, sjá á næstu síðu að þetta er rétt stikun, eða möguleg stikun.

$$\dot{x} = R\{-\omega \sin(\omega t) - (\dot{\phi} + \omega)\sin(\phi + \omega t)\}$$

$$\dot{y} = R\{\omega \cos(\omega t) + (\dot{\phi} + \omega)\cos(\phi + \omega t)\}$$

$$L = T = \frac{m}{2}\{\dot{x}^2 + \dot{y}^2\} = \frac{mR^2}{2}\Big[\omega^2 + (\dot{\phi}+\omega)^2 + 2\omega(\dot{\phi}+\omega) \cdot \big[+\sin(\omega t)\sin(\phi+\omega t) + \cos(\omega t)\cos(\phi+\omega t)\big]\Big]$$



$$L = \frac{mR^2}{2}\{\omega^2 + (\dot{\phi}+\omega)^2 + 2\omega(\dot{\phi}+\omega)\cos\phi\}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \rightarrow -\frac{mR^2}{2} 2\omega(\dot{\phi}+\omega)\sin\phi - \frac{d}{dt}\left\{\frac{mR^2}{2} 2(\dot{\phi}+\omega) + 2\omega\cos\phi \cdot \frac{mR^2}{2}\right\} = 0$$

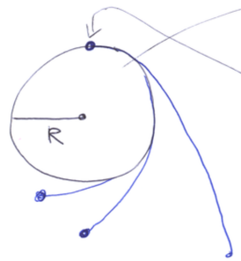
$$\rightarrow -2\omega(\dot{\phi}+\omega)\sin\phi - 2\ddot{\phi} + 2\omega\dot{\phi}\sin\phi = 0$$

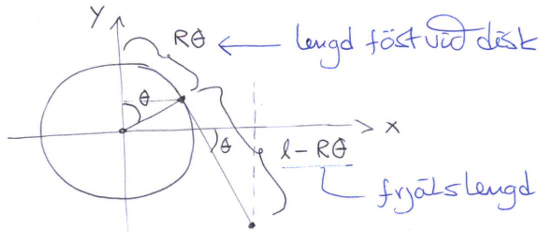
$$\rightarrow \boxed{\ddot{\phi} + \omega^2 \sin\phi = 0}$$

hreyfijafna pendúls

(5) 7-18 í bók

Kyrr diskur, pendúll með lengd l festur við topp hans





$$x = \{l - R\theta\}\cos\theta + R\sin\theta$$

$$y = -\{l - R\theta\}\sin\theta + R\cos\theta$$

$$\dot{x} = -l\dot{\theta}\sin\theta - R\dot{\theta}\cos\theta + R\theta\dot{\theta}\sin\theta + R\dot{\theta}\cos\theta = -l\dot{\theta}\sin\theta + R\theta\dot{\theta}\sin\theta$$

$$\dot{y} = -l\dot{\theta}\cos\theta + R\dot{\theta}\sin\theta + R\theta\dot{\theta}\cos\theta - R\dot{\theta}\sin\theta = -l\dot{\theta}\cos\theta + R\theta\dot{\theta}\cos\theta$$

$$\rightarrow T = \frac{m}{2}\left\{\dot{x}^2+\dot{y}^2\right\} = \frac{m}{2}\left\{(l\dot\theta)^2 + (R\theta\dot\theta)^2 - 2Rl\theta\dot\theta^2\right\}$$

$$U = mgy = mg\left\{R\cos\theta - (l-R\theta)\sin\theta\right\}$$

$$L = \frac{m}{2}\left\{(l\dot\theta)^2+(R\theta\dot\theta)^2 - 2Rl\theta\dot\theta^2\right\} - mg\left\{R\cos\theta-(l-R\theta)\sin\theta\right\}$$

Eitt alhnit θ

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)=0 \rightarrow m(R\dot\theta)^2\theta - mRl\dot\theta^2 + mgR\sin\theta$$
$$- R\sin\theta\cdot mg + mg(l-R\theta)\cos\theta$$
$$- \frac{d}{dt}\left\{ml^2\dot\theta + (R\theta)^2 m\dot\theta - 2Rl\theta\dot\theta m\right\} = 0$$

$$m(R\dot\theta)^2\theta - mRl\dot\theta^2 + mgR\sin\theta - mgR\sin\theta + mg(l-R\theta)\cos\theta$$
$$- ml^2\ddot\theta - (R\theta)^2 m\ddot\theta - R^2 m\dot\theta\, 2\theta\dot\theta + 2mRl\theta\ddot\theta + 2Rl\dot\theta^2 m = 0$$

$$(l-R\theta)^2\ddot\theta - R(l-R\theta)\dot\theta^2 - g(l-R\theta)\cos\theta = 0$$

$$\rightarrow \boxed{(l-R\theta)\ddot\theta - R\dot\theta^2 - g\cos\theta = 0}$$ er hreyfijafnan

Viljum finna (litið) horn θ_0 um hvert smáar sveiflur eru samhverfar

$$\theta = \theta_0 + \delta$$

$\ddot\theta = \ddot\delta$ $\qquad \cos\theta = \cos(\theta_0+\delta) = \cos\theta_0\cdot\cos\delta - \sin\theta_0\cdot\sin\delta$

$\dot\theta = \dot\delta$

Hreyfijafnan verður þá

$$\left\{l - R(\theta_0+\delta)\right\}\ddot\delta - R\underbrace{\dot\delta^2}_{\sim 0} - g\left\{\cos\theta_0\underbrace{\cos\delta}_{\sim 1} - \sin\theta_0\underbrace{\sin\delta}_{\sim\delta}\right\} = 0$$

$$\ddot\delta + \left\{\frac{g\sin\theta_0}{l-R\theta_0}\right\}\delta = \left\{\frac{g\cos\theta_0}{l-R\theta_0}\right\}$$

Almenna lausnin fyrir ohliðraða jöfnuna er

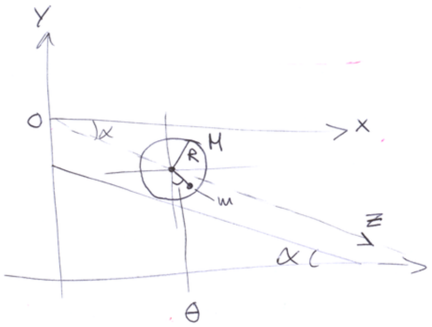
$\delta(t) = A\sin(\omega t+\phi_0)$ þar sem

$$\omega = \sqrt{\frac{g\sin\theta_0}{l-R\theta_0}}$$

Sérlausn má lesa úr jöfnunni $\frac{\cos\theta_0}{\sin\theta_0}$

$\rightarrow$ Sveiflan er samhverf um θ_0 þegar $\theta_0 = \frac{\pi}{2}$ því þá er lausnin ð eins,

(6) Dæmi 7-09 í bók



pendúll festur á ás disksins með lengd $l < R$ og massa m

Notum alhnitin z og θ

CM-disks:

$x = z\cos\alpha$

$y = -z\sin\alpha$

$$T = \frac{M}{2}\dot z^2 + \frac{I}{2}\dot\phi^2$$

$z = R\phi$

↑ veltiskilyrði

$U = +Mgy$

massi pendúls

$x_b = z\cos\alpha + l\sin\theta$

$y_b = -z\sin\alpha - l\cos\theta$

$$T = \frac{m}{2}\left\{\dot x_b^2 + \dot y_b^2\right\}$$

$$U = -mgz\sin\alpha - mgl\cos\theta$$

Ihvilket

$$T = \frac{M+m}{2}\dot{z}^2 + \frac{I}{2}\dot{\phi}^2 + m\ell^2\dot{\theta}^2 + m\ell\dot{\theta}\dot{z}\cos(\theta+\alpha)$$

$$U = -(M+m)gz\sin\alpha - mg\ell\cos\theta$$

og $I = \frac{MR^2}{2}$, $z = R\phi$

$$\rightarrow L = T-U = \left\{\frac{3M}{4} + \frac{m}{2}\right\}\dot{z}^2 + \frac{m}{2}(\ell\dot{\theta})^2 + m\ell\dot{\theta}\dot{z}\cos(\theta+\alpha)$$
$$-(M+m)gz\sin\alpha + mg\ell\cos\theta$$

allmittin ove z og θ

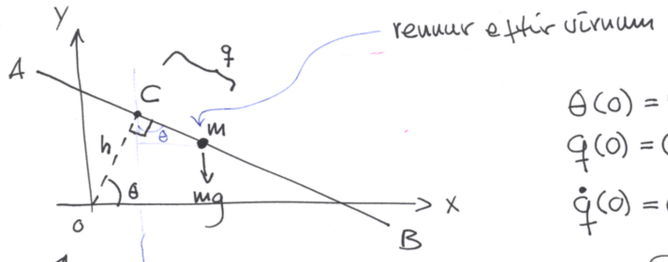
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \rightarrow \boxed{\ddot{\theta} + \frac{\ddot{z}\cos(\theta+\alpha)}{\ell} + \frac{g}{\ell}\sin\theta = 0}$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \rightarrow$$

$$\left\{\frac{3M}{2} + m\right\}\ddot{z} - (M+m)g\sin\alpha + m\ell\left\{\ddot{\theta}\cos(\theta+\alpha) - \dot{\theta}^2\sin(\theta+\alpha)\right\} = 0$$

(1)

(1) Dæmi 7-17 í bók



$\theta(0) = 0$

$q(0) = 0$

$\dot q(0) = 0$

Sýna að $q(t) = \frac{g}{2\omega^2}\{\cosh(\omega t) - \cos(\omega t)\}$

Snýst í sléttunni $\dot\theta = \omega$ = fasti um O

$x = h\cos\theta + q\sin\theta \qquad \theta = \omega t$

$y = h\sin\theta - q\cos\theta \qquad q = q(t)$

$\rightarrow \quad \dot x = -h\omega\sin(\omega t) + q\omega\cos(\omega t) + \dot q\sin(\omega t)$

$\dot y = h\omega\cos(\omega t) + q\omega\sin(\omega t) - \dot q\cos(\omega t)$

(2)

$\rightarrow T = \frac{m}{2}\{\dot x^2 + \dot y^2\} = \frac{m}{2}\{(h\omega)^2 + (q\omega)^2 + \dot q^2 - 2h\omega\dot q\}$

allir aðrir „blandaðir liðir" hverfa vegna formerkja

$U = mgy = mg\{h\sin(\omega t) - q\cos(\omega t)\}$

því fæst

$L = \frac{m}{2}\{(h\omega)^2 + (q\omega)^2 + \dot q^2 - 2h\omega\dot q\} - mg\{h\sin(\omega t) - q\cos(\omega t)\}$

Aðeins eitt alhnit, q

$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = 0 \rightarrow m\omega^2 q + mg\cos(\omega t) - \frac{d}{dt}\{m\dot q - mh\omega\} = 0$

$\rightarrow \boxed{\ddot q - \omega^2 q = g\cos(\omega t)}$

(3)

„Samskonar" jafna og í 1. dæmi á síðasta dæmablaði

$q(t) = Ae^{\omega t} + Be^{-\omega t} - \frac{g}{2\omega^2}\cos(\omega t)$

Athugum upphafsskilyrðin

$q(0) = A + B - \frac{g}{2\omega^2} = 0$

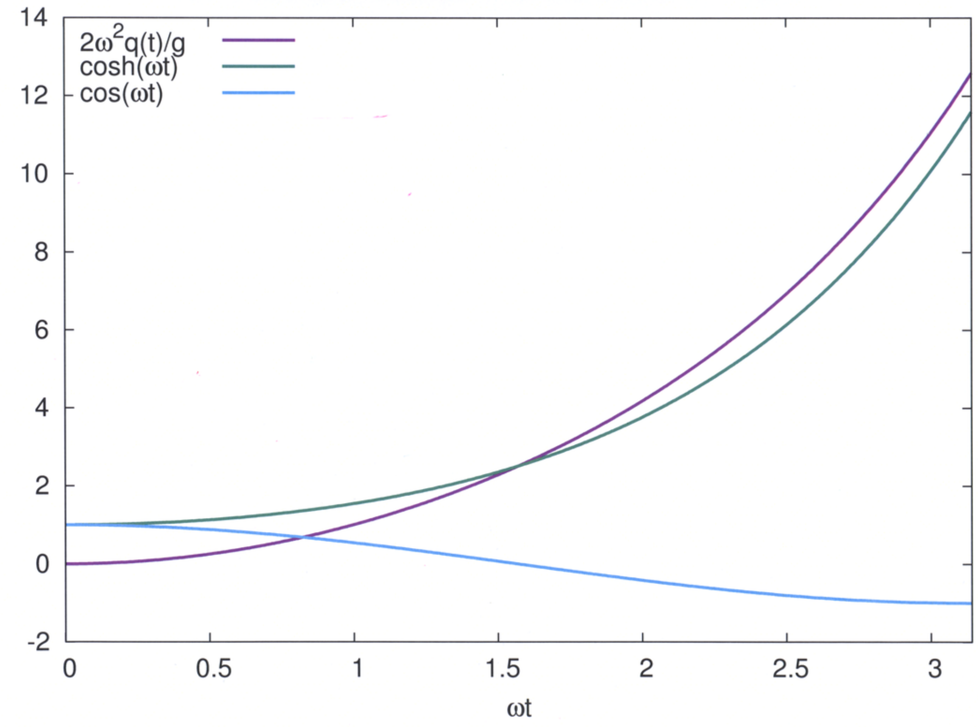
$\dot q(0) = A - B = 0$

$\rightarrow \quad A = \frac{g}{4\omega^2}, \quad B = -\frac{g}{4\omega^2}$

$\rightarrow \quad q(t) = \frac{g}{2\omega^2}\{\cosh(\omega t) - \cos(\omega t)\}$

sjá mynd á næstu síðu af q(t).

(4)



(5)

finnum fall Hamiltons

$$p = \frac{\partial L}{\partial \dot q} = m\dot q - mh\omega \;\rightarrow\; \dot q = \frac{p}{m} + h\omega$$

$$H = p\dot q - L = m\dot q^2 - mh\omega \dot q - L$$

$$= \frac{m}{2}\left\{\dot q^2 - (h\omega)^2 - (q\omega)^2\right\} + mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\}$$

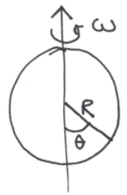
$$= \frac{p^2}{2m} + ph\omega - \frac{1}{2}m(q\omega)^2 + mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\}$$

$$E = T + U = \frac{p^2}{2m} + \frac{1}{2}m(q\omega)^2 + mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\} \neq H$$

$H = H(t)$ og því ekki heildarorkan, snúningur í þyngdarsviði leiðir til orkutilfærslu

(6)

③ Dæmi 7-21 í bók



Eitt alhnit, θ

snúningshreyfing með ferð ω

$$T = \frac{m}{2}(R\dot\theta)^2 + \frac{m}{2}(R\sin\theta)^2\omega^2, \quad U = -mgR\cos\theta$$

hreyfing á hring

Finna jafnvægisstöðu einder og smáar sveiflur um hana

Fyrst, hreyfijafna

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0 \;\rightarrow\; m(R\omega)^2\sin\theta\cdot\cos\theta - mgR\sin\theta - \frac{d}{dt}\left\{mR^2\dot\theta\right\} = 0$$

$$\rightarrow \boxed{\ddot\theta - \omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0}$$

(7)

Jafnvægisstöður eru þegar „kraftar jafnast út" eða þess vegna þegar hröðun er hverfandi

þegar $\ddot\theta\big|_{\theta_i} = 0$ Hreyfijafnan gefur

$$\ddot\theta\big|_{\theta_i} = \left\{\omega^2\cos\theta_i - \frac{g}{R}\right\}\sin\theta_i = 0$$

Sem hefur 3 lausnir

$$\theta_1 = 0, \quad \theta_2 = \pi, \quad \theta_3 = \arccos\left\{\frac{g}{\omega^2 R}\right\}$$

Skilgreinum $\delta_i = \theta - \theta_i$ og fáum hreyfijöfnu

$$\ddot\delta_i - \omega^2\left\{\cos(\theta_i + \delta_i) - \frac{g}{\omega^2 R}\right\}\sin(\theta_i + \delta_i) = 0$$

(8)

$$\sin(\theta_i + \delta_i) = \sin\theta_i\cdot\cos\delta_i + \cos\theta_i\cdot\sin\delta_i$$
$$\approx \sin\theta_i + \delta_i\cos\theta_i$$

$$\cos(\theta_i + \delta_i) = \cos\theta_i\cdot\cos\delta_i - \sin\theta_i\sin\delta_i$$
$$\simeq \cos\theta_i - \delta_i\sin\theta_i$$

því er hreyfijafnan

$$\ddot\delta_i - \omega^2\left\{\cos\theta_i - \frac{g}{\omega^2 R} - \delta\sin\theta_i\right\}\left\{\sin\theta_i + \delta_i\cos\theta_i\right\} \approx 0$$

$\theta_1 = 0$

$$\rightarrow \boxed{\ddot\delta_1 - \omega^2\left\{1 - \frac{g}{\omega^2 R}\right\}\delta = 0}$$

stöðugt ef $\omega^2 < \frac{g}{R}$ með tíðni $\omega_1 = \sqrt{\omega^2 - \frac{g}{R}}$

(9)

$\theta_2 = \pi$

$$\rightarrow \ddot{\delta}_2 - \omega^2\left\{1 + \frac{g}{\omega^2 R}\right\}\delta_i = 0$$

sem er alltaf óstöðug „sveifla" því $\omega_2^2 = -\left\{\omega^2 + \frac{g}{R}\right\} < 0$

$\theta_3 = \arccos\left\{\frac{g}{\omega^2 R}\right\}$

$$\rightarrow \ddot{\delta}_3 + \omega^2 \sin\theta_3 \cdot \delta_3 \cdot \sin\theta_3 = 0$$

$$\ddot{\delta}_3 + \omega^2 \sin^2\theta_3 \cdot \delta_3 = 0$$

$$\sin^2\theta_3 = 1 - \cos^2\theta_3 = 1 - \frac{g^2}{R^2\omega^4}$$

$$\rightarrow \omega_3 = \sqrt{\omega^2 - \left(\frac{g}{R\omega}\right)^2} = \omega^2\sqrt{1 - \left(\frac{g}{R\omega^2}\right)^2}$$

(10)

skilgreinum marktíðni $\omega_c^2 = \frac{g}{R}$

$$\rightarrow \omega_1 = \sqrt{\omega^2 - \omega_c^2} = \omega\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$ stöðug $\omega < \omega_c$

$$\omega_3 = \sqrt{\omega^2 - \frac{\omega_c^4}{\omega^2}} = \omega\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^4}$$ stöðug $\omega \geq \omega_c$

Finnum H

$$H = p\dot{q} - L = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L$$

$$= mR^2\dot{\theta}^2 - \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}(\omega R\sin\theta)^2 + mgR\cos\theta$$

$$= \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}(\omega R\sin\theta)^2 + mgR\cos\theta$$

$$\rightarrow \frac{H}{mR^2} = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}(\omega\sin\theta)^2 + \frac{g}{R}\cos\theta$$

$$\rightarrow \frac{H}{mR^2\omega_c^2} = \frac{1}{2}\left\{\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\sin\theta\right)^2\right\} - \cos\theta$$

(11)

því er fasarit fyrir fast $\frac{H}{mR^2\omega_c^2} = \kappa$

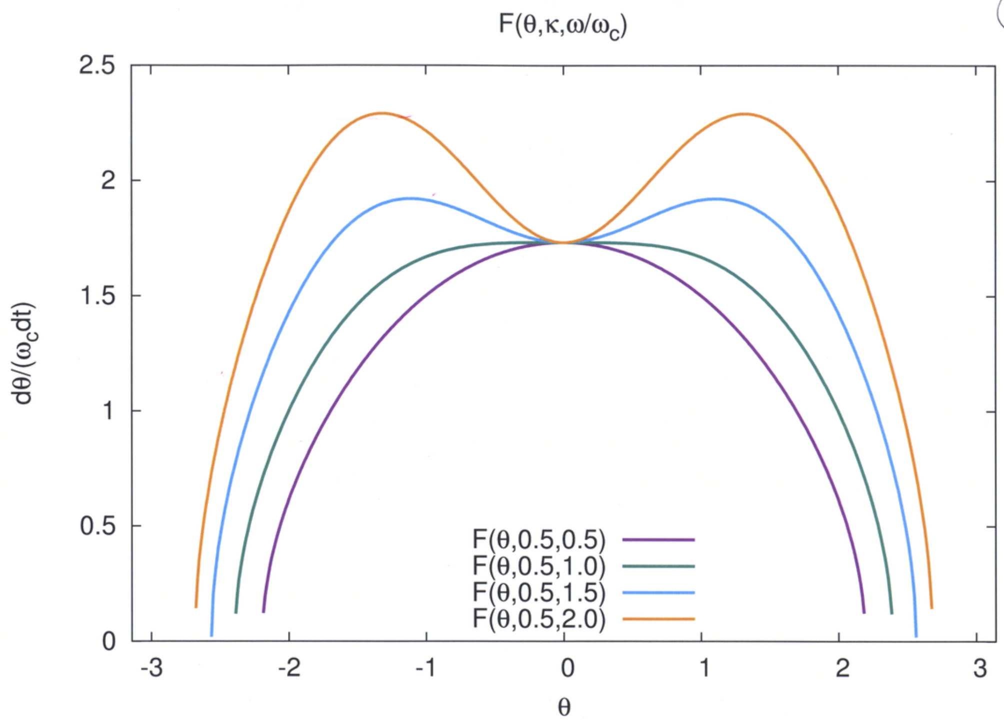
$$\left(\frac{\dot{\theta}}{\omega_c}\right)^2 = 2\{\kappa + \cos\theta\} + \left(\frac{\omega}{\omega_c}\right)^2\sin^2\theta$$

$$\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\sin\theta\right)^2 = 2\{\kappa + \cos\theta\}$$

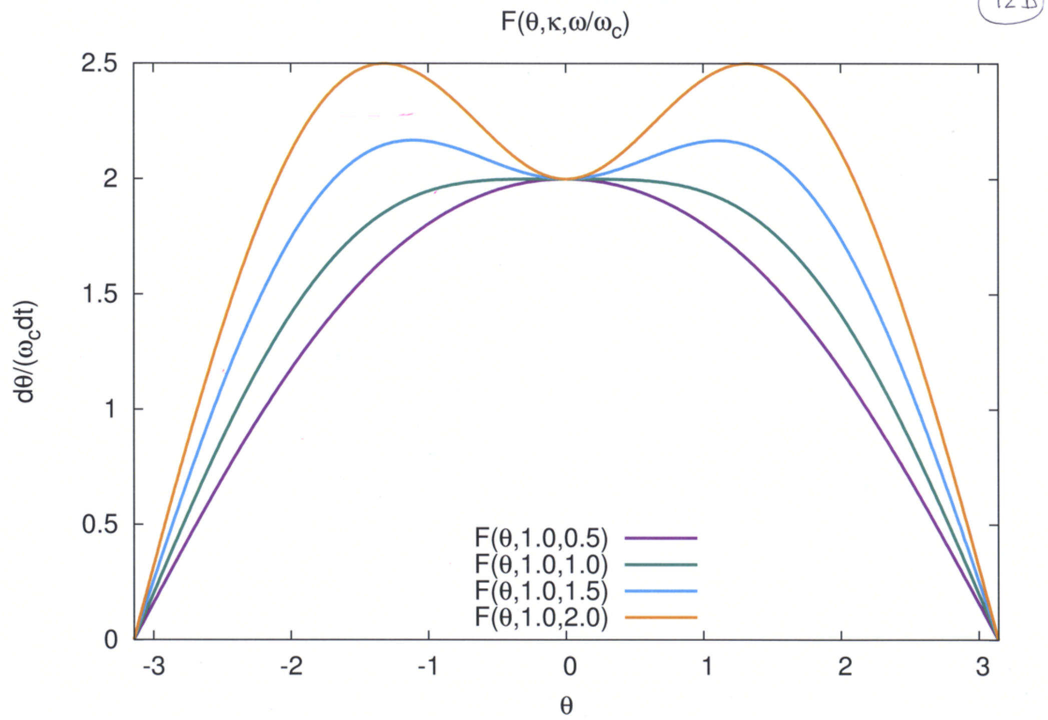
$$\left(\frac{\dot{\theta}}{\omega_c}\right) = \sqrt{2\{\kappa + \cos\theta\} + \left\{\frac{\omega}{\omega_c}\sin\theta\right\}^2} = F\left(\theta, \kappa, \frac{\omega}{\omega_c}\right)$$

Sjá gröf...

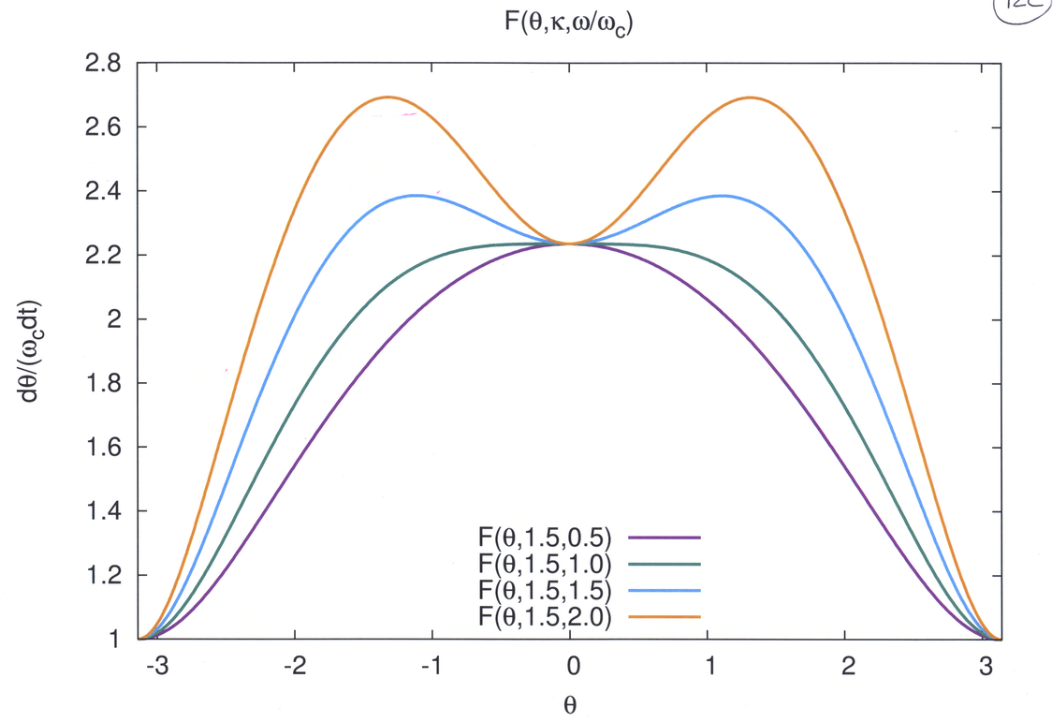
(12a)



12b



12c



13

(2) Einvíður sveifill með $U = \lambda x^4$ | $\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$

$T = \frac{m}{2}\dot{x}^2$, $L = \frac{m}{2}\dot{x}^2 - \lambda x^4$ | $-4\lambda x^3 - m\ddot{x} = 0$

$\rightarrow \boxed{\ddot{x} + \frac{4\lambda}{m}x^3 = 0}$

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$H = p\dot{x} - L = m\dot{x}^2 - \frac{m}{2}\dot{x}^2 + \lambda x^4 = \frac{m}{2}\dot{x}^2 + \lambda x^4$

$= \frac{p^2}{2m} + \lambda x^4$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$-\dot{p} = \frac{\partial H}{\partial x} = 4\lambda x^3$$

hreyfijöfnur Hamiltons fyrir kerfið

14

(4) <u>Dæmi 7-22 í bók</u>

$F(x,t) = \frac{k}{x^2} e^{-\frac{t}{\tau}}$ Kraftur á ögn

finna L og H og staða varðveislu orku

$U = \frac{k}{x} e^{-t/\tau}$ því $F = -\frac{\partial U}{\partial x}$

$\boxed{L = \frac{m}{2}\dot{x}^2 - \frac{k}{x}e^{-t/\tau}}$, $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

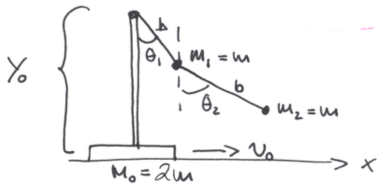
$H = p\dot{x} - L = m\dot{x}^2 - \frac{m}{2}\dot{x}^2 + \frac{k}{x}e^{-t/\tau}$

$= \frac{m}{2}\dot{x}^2 + \frac{k}{x}e^{-t/\tau} = \boxed{\frac{p^2}{2m} + \frac{k}{x}e^{-t/\tau}}$

$H = T + U \neq$ fasti

því $U \neq U(\dot{x})$

⑤ Dæmi 7-40 í bók



m_0: x, $Y = 0$ } Jafn hæði breytir engu hér

m_1: $x_1 = x + b\sin\theta_1$, $y_1 = y_0 - b\cos\theta_1$

m_2: $x_2 = x_1 + b\sin\theta_2$, $y_2 = y_1 - b\cos\theta_2$

$$\dot{x}_1 = \dot{x} + b\cos\theta_1\cdot\dot{\theta}_1$$
$$\dot{y}_1 = b\sin\theta_1\cdot\dot{\theta}_1$$
$$\dot{x}_2 = \dot{x} + b\cos\theta_1\cdot\dot{\theta}_1 + b\cos\theta_2\cdot\dot{\theta}_2$$
$$\dot{y}_2 = b\sin\theta_1\cdot\dot{\theta}_1 + b\sin\theta_2\cdot\dot{\theta}_2$$

$$U = -mgb\cos\theta_1 - mg\{b\cos\theta_1 + b\cos\theta_2\}$$

Þar sem við látum við með y_0 hverfa

$$T = \frac{2m}{2}\dot{x}^2 + \frac{m}{2}\left\{\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right\}$$
$$= m\dot{x}^2 + \frac{m}{2}\left\{\dot{x}^2 + (b\dot{\theta}_1)^2 + 2b\cos\theta_1\cdot\dot{x}\dot{\theta}_1\right\}$$
$$+ \frac{m}{2}\left\{\left[\dot{x} + b\dot{\theta}_1\cos\theta_1 + b\dot{\theta}_2\cos\theta_2\right]^2 + \left[b\dot{\theta}_1\sin\theta_1 + b\dot{\theta}_2\sin\theta_2\right]^2\right\}$$

$$L = 2m\dot{x}^2 + mb^2\dot{\theta}_1^2 + 2mb\dot{x}\dot{\theta}_1\cos\theta_1 + \frac{m}{2}mb^2\dot{\theta}_2^2 + mb\dot{x}\dot{\theta}_2\cos\theta_2$$
$$+ mb^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + 2mgb\cos\theta_1 + mgb\cos\theta_2$$

3 breytur, x, θ_1, og θ_2

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \to \frac{d}{dt}\left\{4m\dot{x} + 2mb\dot{\theta}_1\cos\theta_1 + mb\dot{\theta}_2\cos\theta_2\right\} = 0$$

Eða

$$4m\ddot{x} + bm\left\{2\ddot{\theta}_1\cos\theta_1 + \ddot{\theta}_2\cos\theta_2\right\} - bm\left\{2\dot{\theta}_1^2\sin\theta + \dot{\theta}_2^2\sin\theta_2\right\} = 0$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 0 \to -2mb\dot{x}\dot{\theta}_1\sin\theta_1 - mb^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)$$
$$- 2mgb\sin\theta_1 - \frac{d}{dt}\left\{mb^2 2\dot{\theta}_1 + 2mb\dot{x}\cos\theta_1 + mb^2\dot{\theta}_2\cos(\theta_1-\theta_2)\right\} = 0$$

$$0 = -2mb\dot{x}\dot{\theta}_1\sin\theta_1 - mb^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - 2mgb\sin\theta_1$$
$$- mb^2 2\ddot{\theta}_1 - 2mb\ddot{x}\cos\theta_1 + 2mb\dot{x}\dot{\theta}_1\sin\theta_1 - mb^2\ddot{\theta}_2\cos(\theta_1-\theta_2)$$
$$+ mb^2\dot{\theta}_2\dot{\theta}_1\sin(\theta_1-\theta_2) - mb^2\dot{\theta}_2^2\sin(\theta_1-\theta_2)$$

$$\to -2mgb\sin\theta_1 - mb^2 2\ddot{\theta}_1 - 2mb\ddot{x}\cos\theta_1 - mb^2\ddot{\theta}_2\cos(\theta_1-\theta_2)$$
$$- mb^2\dot{\theta}_2^2\sin(\theta_1-\theta_2) = 0$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = 0$$

$$\to -mg\sin\theta_2 - mb\ddot{\theta}_2 + m\ddot{x}\cos\theta_2 + mb\ddot{\theta}_1\cos(\theta_1-\theta_2) - mb\dot{\theta}_1^2\sin(\theta_1-\theta_2) = 0$$

⑥ Dæmi 7-30 í bók

a) $$[g,h] \equiv \sum_k \left\{\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k}\right\}$$ gefið

Finna $$\frac{dg}{dt} = [g,H] + \frac{\partial g}{\partial t}$$

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left\{\frac{\partial g}{\partial q_k}\frac{\partial q_k}{\partial t} + \frac{\partial g}{\partial p_k}\frac{\partial p_k}{\partial t}\right\}$$

nota síðan

$$\frac{\partial q_k}{\partial t} = \dot{q} = \frac{\partial H}{\partial p_k} \quad \text{og} \quad \frac{\partial p_k}{\partial t} = \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Jöfnur Hamiltons

$$\rightarrow \frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}$$

(3) Dæmi 8-02 í bók.

Heildið

$$\theta(r) = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu\left\{E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right\}}} + C \qquad \leftarrow \text{fasti}$$

Í bókinni er bent á breytuskipti $u = \frac{1}{r}$, $du = -\frac{1}{r^2}dr$

$$\rightarrow \theta = -\int \frac{du\cdot\ell}{\sqrt{2\mu\left\{E + uk - \frac{u^2\ell^2}{2\mu}\right\}}} = -\int \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu k}{\ell^2}u - u^2}}$$

Notum (E.8b) eða (GR-2.261)

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{-a}}\arcsin\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right) \qquad \begin{cases} a<0 \\ b^2 > 4ac \\ |2ax+b| < \sqrt{b^2-4ac} \end{cases}$$

$$\rightarrow \theta + \theta_0 = + \arcsin\left[\frac{-2u + \frac{2\mu k}{\ell^2}}{\sqrt{\left(\frac{2\mu k}{\ell^2}\right)^2 + \frac{8\mu E}{\ell^2}}}\right]$$

$$\rightarrow \sin(\theta+\theta_0) = \frac{-\frac{2}{r} + \frac{2\mu k}{\ell^2}}{\sqrt{\left(\frac{2\mu k}{\ell^2}\right)^2 + \frac{8\mu E}{\ell^2}}}$$

heildunarfasti (θ_0)

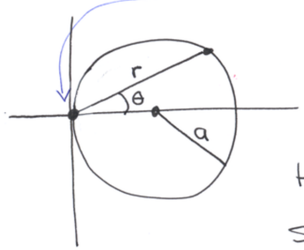
Notum $\sin(\theta - \frac{\pi}{2}) = -\cos\theta$, þ.e. veljum $\theta_0 = -\frac{\pi}{2}$

$$\rightarrow \cos\theta = \frac{-\frac{2\mu k}{\ell^2} + \frac{2}{r}}{\sqrt{\left(\frac{2\mu k}{\ell^2}\right)^2 + \frac{8\mu E}{\ell^2}}} = \frac{\frac{\ell^2}{\mu k r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}$$

(2) Dæmi 8.11 í bók

"Ögn í kraftsviði $F(r) = -\frac{k}{r^n}$

Ef brautin er hringbraut í gegnum kraftmiðju, sýnið að $n = 5$



Hreyfijafnan (8.21) er

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

Hægt er að stika hringinn miðað við kraftmiðju sem $r = 2a\cos\theta$, sú braut ætti að gefa kraftinn, reynum

$$\frac{1}{r} = \frac{1}{2a\cos\theta} \rightarrow \frac{d^2}{d\theta^2}\left(\frac{1}{2a\cos\theta}\right) = \frac{\sin^2\theta}{a\cos^3\theta} + \frac{1}{2a\cos\theta}$$

(þægilegt með maxima)

Setjum inn í hreyfijöfnu

$$\rightarrow \frac{\sin^2\theta}{a\cos^3\theta} + \frac{1}{2a\cos\theta} + \frac{1}{2a\cos\theta} = -\frac{\mu}{\ell^2}\left\{(2a)^2\cos^2\theta\right\}F$$

$$\rightarrow \frac{\sin^2\theta}{\cos^5\theta} + \frac{1}{\cos^3\theta} = -\frac{\mu}{\ell^2}4a^3 F$$

$$\rightarrow \frac{\sin^2\theta + \cos^2\theta}{\cos^5\theta} = -\frac{\mu}{\ell^2}4a^3 F$$

$$\rightarrow \frac{1}{\cos^5\theta} = -\frac{\mu}{\ell^2}4a^3 F \rightarrow F = -\frac{\ell^2}{4a^3\mu}\frac{1}{\cos^5\theta}$$

$$\rightarrow \boxed{F = -\frac{\ell^2}{4a^3\mu}\frac{(2a)^5}{\{2a\cos\theta\}^5} = -\frac{8a^2\ell^2}{\mu}\frac{1}{r^5}}$$

(5)

① Dæmi 8-08 í bók

"Ögn er hrint frá kraftmiðju með $F(r) = kr$

$\bar{F} = -\nabla U \rightarrow U = -\frac{kr^2}{2}$, sýna að brautin sé gleiðbogi.

Lausn hreyfijöfnunnar er (8.17)

$$\theta(r) = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}} = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu\left(E + \frac{kr^2}{2} - \frac{\ell^2}{2\mu r^2}\right)}}$$

Reiknum breytuskipti $x = r^2 \rightarrow dx = 2r\,dr$, $\frac{dr}{r^2} = \frac{dx}{r^2 2r} = \frac{1}{2}\frac{dx}{x}\frac{1}{\sqrt{x}}$

$$\rightarrow \theta = \frac{1}{2}\int \frac{dx}{x\sqrt{\frac{2\mu E}{\ell^2}x + \frac{\mu k}{\ell^2}x^2 - 1}} = \frac{1}{2}\int \frac{dx}{x\sqrt{\frac{\mu k}{\ell^2}x^2 + \frac{2\mu E}{\ell^2}x - 1}}$$

(6)

Heildið er leysanlegt skv. (E.10b) eða (GR-2.266)

$$\theta = \frac{1}{2}\arcsin\left\{\frac{\frac{\mu E}{\ell^2}x - 1}{x\sqrt{\left(\frac{\mu E}{\ell^2}\right)^2 + \frac{\mu k}{\ell^2}}}\right\} + \theta_0$$

$$\rightarrow \sin\{2(\theta-\theta_0)\} = \frac{\frac{\mu E}{\ell^2}}{\sqrt{\left(\frac{\mu E}{\ell^2}\right)^2 + \frac{\mu k}{\ell^2}}} - \frac{1}{r^2\sqrt{\left(\frac{\mu E}{\ell^2}\right)^2 + \frac{\mu k}{\ell^2}}}$$

$$= \frac{1}{\sqrt{1 + \frac{k\ell^2}{\mu E^2}}} - \frac{1}{r^2}\frac{\frac{\ell^2}{\mu E}}{\sqrt{1 + \frac{k\ell^2}{\mu E^2}}} \equiv \alpha - \frac{\beta}{r^2}$$

$\sin\{2(\theta-\theta_0)\} = \alpha - \frac{\beta}{r^2}$, Ef við hefðum $\cos(2\theta)$ þá mætti nota $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

Setjum því $\theta_0 = \frac{\pi}{4}$

(7)

$$\sin\left\{2\theta - \frac{\pi}{2}\right\} = -\cos(2\theta) = \alpha - \frac{\beta}{r^2} \rightarrow r^2\cos(2\theta) + r^2\alpha = \beta$$

$$\rightarrow r^2\{\cos^2\theta - \sin^2\theta\} + r^2\alpha = \beta$$

$$x^2 - y^2 + (x^2+y^2)\alpha = \beta$$

$$\boxed{x^2(1+\alpha) - y^2(1-\alpha) = \beta}$$

$\alpha = \frac{1}{\sqrt{1 + \frac{k\ell^2}{\mu E^2}}} < 1$

$\rightarrow 1 - \alpha > 0$

→ svo þetta er jafna gleiðboga

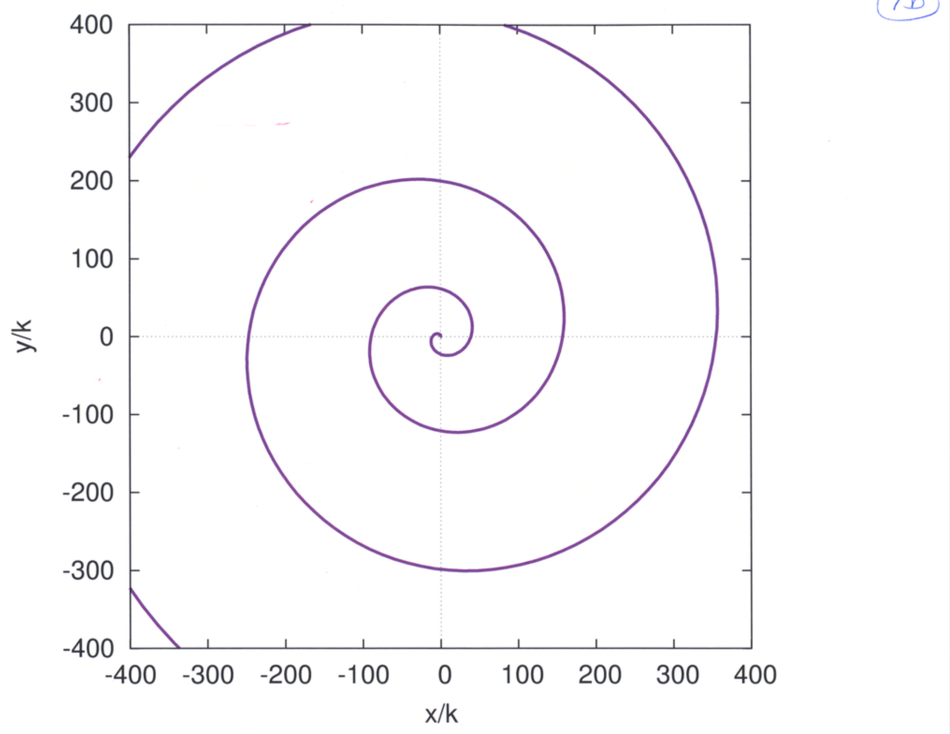
④ Dæmi 8-14 í bók

Braut gefin sem $r = k\theta^2$ (fasti)

Finna kraftinn

sjá mynd á næstu síðu

(7b)



Notum hreyfijöfnu (8.21)

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\frac{1}{r} = \frac{1}{k\theta^2} \;\rightarrow\; \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{6k}{r^2}$$

$$\rightarrow \frac{6k}{r^2} + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$-\left\{\frac{6k}{r^4} + \frac{1}{r^3}\right\}\frac{\ell^2}{\mu} = F(r)$$

(5) Dæmi 8-22 í bók

Skoðum hreyfingu í krafti

$$F(r) = -\frac{k}{r^3} \;\rightarrow\; U(r) = -\frac{k}{2r^2}$$

Virka mættið

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$= \frac{1}{2}\left\{\frac{\ell^2}{\mu} - k\right\}\frac{1}{r^2}$$

miðflótta kraftur og mættið hafa sama form

Schrödinger - - - - -

Athugum hreyfijöfnuna

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2u^2}F(1/u) = -\frac{\mu}{\ell^2u^2}\left\{-ku^3\right\} = +\frac{\mu k}{\ell^2}u$$

$$\rightarrow \frac{d^2u}{d\theta^2} + \left\{1 - \frac{\mu k}{\ell^2}\right\}u = 0$$

skoðum tilfellin sem koma til greina

(1) $\ell^2 = \mu k$ (ℓ tengist hverfiþunga)

$$\rightarrow \frac{d^2u}{d\theta^2} = 0 \quad \text{með lausn} \quad u = A\theta + B$$

$$u = \frac{1}{r} \;\rightarrow\; r = \frac{1}{A\theta + B}$$

gormhreyfing að miðju

(2) $\ell^2 > \mu k \;\rightarrow\; \left\{1 - \frac{\mu k}{\ell^2}\right\} = \gamma^2 > 0$

$$\rightarrow \frac{d^2u}{d\theta^2} + \gamma^2 u = 0$$

með lausn $u = A\cos(\gamma\theta + \delta) \;\rightarrow\; r = \frac{1}{A\cos(\gamma\theta+\delta)}$

$u \in [-1,1] \;\rightarrow\; Ar \in \mathbb{R}$ og brautin er opin

(3) $\ell^2 < \mu k \;\rightarrow\; \left\{1 - \frac{\mu k}{\ell^2}\right\} = -\kappa^2$, með $\kappa^2 > 0$

$$\rightarrow \frac{d^2u}{d\theta^2} - \kappa^2 u = 0 \;\rightarrow\; u = A\cosh(\kappa\theta + \delta)$$

$$r = \frac{1}{A\cosh(\kappa\theta+\delta)} \qquad \cosh x \in [1,\infty)$$

$\rightarrow Ar \in (0,1]$ hreyfing að miðju

Stöðugleiki hringbrautar

$$g(r) = \frac{1}{\mu}\frac{\partial U}{\partial r} = \frac{k}{\mu r^3}, \qquad r \rightarrow \rho + x \quad \text{þ.s.} \quad x \ll \rho$$

frávik frá hring

$$\ddot{x} - \frac{\ell^2}{\mu^2\rho^3\left[1+\frac{x}{\rho}\right]^3} = -g(\rho + x)$$

$$= -\frac{k}{\mu\rho^3\left[1+\frac{x}{\rho}\right]^3}$$

$$\rightarrow \ddot{x} + \left\{k - \frac{\ell^2}{\mu}\right\}\frac{1}{\mu\rho^3\left(1+\frac{x}{\rho}\right)^3} = 0$$

$$g(\rho) = \frac{\ell^2}{\mu^2\rho^3} \;\rightarrow\; k = \frac{\ell^2}{\mu}$$

$\ddot{r}\big|_{r=\rho} = 0 \;\rightarrow\; k = \frac{\ell^2}{\mu}$ fyrir stöðuga hringbraut

$\rightarrow \ddot{x} = 0 \;\leftarrow$ en þessi jafna gefur ekki lotubundna lausn $\rightarrow$ engin stöðug hringbraut

(6) Dæmi 8-21 í bók

E = fasti

Ef stöðug hringbraut þá er l^2 stærra en fyrir fyrir aðra braut

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$

$$\longrightarrow \; l^2 = 2\mu r^2 \left\{ E - U(r) - \frac{\mu}{2}\dot{r}^2 \right\}$$

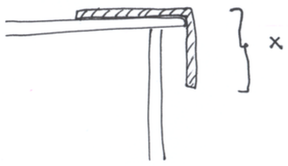
Sama E og U, hringbraut $\longrightarrow \dot{r} = 0$

fyrir allar aðrar brautir $\dot{r} \neq 0$

$\longrightarrow$ þessi liður dregur úr l^2

(1)

(1) Dæmi 9-21 í bók



L : heildar lengd 1 m

$x(0) = 30\,\text{cm} = x_0$

$v(0) = 0$

finna τ þegar endi reipis fer yfir brúnina á borðinu.

Massi reipis: m

massi reipis fram yfir brún : $\frac{x}{L} \cdot m$

$$\rightarrow m\ddot{x} = m\frac{x}{L} \cdot g \quad \rightarrow \text{hreyfijafna: } \ddot{x} - \frac{x}{L}g = 0$$

Lausn er þá $x = Ae^{\omega t} + Be^{-\omega t}$

$$\rightarrow \ddot{x} = \omega^2 x \quad \rightarrow \omega^2 = \frac{g}{L} \rightarrow \omega = \sqrt{\frac{g}{L}}$$

Finnum A og B til að uppfylla upphafsskilyrði

(2)

$$x(0) = x_0 = A + B, \quad \dot{x}(0) = 0 = A\omega - B\omega \quad \rightarrow \quad A = \frac{x_0}{2}, \quad B = \frac{x_0}{2}$$

$$\rightarrow x(t) = x_0 \operatorname{Cosh}(\omega t)$$

$$L = x_0 \operatorname{Cosh}(\omega\tau) \rightarrow \frac{L}{x_0} = \operatorname{Cosh}(\omega\tau)$$

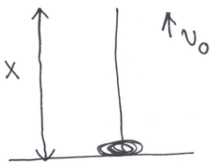
Þá $\tau = \frac{1}{\omega}\operatorname{ArCosh}\left(\frac{L}{x_0}\right) = \sqrt{\frac{L}{g}}\operatorname{ArCosh}\left(\frac{L}{x_0}\right)$

$$\tau = \sqrt{\frac{1.0}{9.81}}\operatorname{ArCosh}\left(\frac{1{,}0}{0{,}3}\right) \approx 0{,}6\ \text{s}$$

(2) Dæmi 9-44 í bók

Reipi með lengd b og massaþéttleika μ liggur vafið upp á borði. Endi er lyft með hraða v_0 finna kraft á hönd þegar a af reipi er á lofti, $a < b$.

(3)



Þyngdar kraftur: $F_g = (\mu x)g$

$$F_{imp} = \frac{d}{dt}\{(\mu x)v\} = \left(\mu\frac{dx}{dt}\right)v + \mu x\frac{dv}{dt}, \quad \frac{dv}{dt} = 0$$

atlagið

$$= \mu v v = \mu v_0^2$$

$\rightarrow$ heildar kraftur $F(x) = \mu v_0^2 + \mu x g$

$$\rightarrow F(a) = \mu\{v_0^2 + ag\} = \mu a g\left\{1 + \frac{v_0^2}{ag}\right\}$$

vegna þyngdar — viðbótin vegna atlagsins

(3) Dæmi 9-41 í bók

Gúmmí bolti fellur úr h_1, kemur aftur upp í hæð h_2

finna ϵ (coeff. of restitution) $\epsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|}$ (teljari: eftir, nefnari: fyrir)

(4)

fall hreyfing $y = v_0 t - \frac{1}{2}gt^2$ og $v = v_0 - gt$

Fyrir árekstur $u_1(0) = 0$, h_1 gefin

$$u_1 = -gt_1 \quad \text{og} \quad h_1 = \frac{1}{2}gt_1^2 \rightarrow t_1 = \sqrt{\frac{2h_1}{g}}$$

$$\rightarrow u_1 = -g\sqrt{\frac{2h_1}{g}} = -\sqrt{2h_1 g}$$

Eftir árekstur

$v_1(0)$: óþekktur

$$v_1(t_2) = 0 = v_1(0) - gt_2 \rightarrow t_2 = \frac{v_1(0)}{g}$$

$$\rightarrow h_2 = v_1(0)t_2 - \frac{1}{2}gt_2^2 = \frac{(v_1(0))^2}{g} - \frac{1}{2}g\frac{(v_1(0))^2}{g^2}$$

$$= \frac{(v_1(0))^2}{2g} \rightarrow v_1(0) = \sqrt{2h_2 g}$$

$$\to \epsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|} = \frac{\sqrt{2h_2 g}}{\sqrt{2h_1 g}} = \sqrt{\frac{h_2}{h_1}}$$

Hreyfiorka töpuð $\qquad \Delta T = T_i - T_f = \left(u_1^2 - (v_1(0))^2\right) m$

$$T_i = m u_1^2$$

$$\Rightarrow \frac{\Delta T}{T_i} = \frac{u_1^2 - (v_1(0))^2}{u_1^2} = 1 - \left(\frac{(v_1(0))^2}{u_1^2}\right) = 1 - \frac{h_2}{h_1}$$

$$= 1 - \epsilon^2$$

(4) Dæmi 9-06 í bók $\qquad \bar{r}_1(0) = 0, \ \bar{r}_2(0) = 0$

Tveir massar m_1 og m_2, með $m_2 = m_1 = m$

$\bar{F}_1 = 0$ og $\bar{F}_2 = F_0 \hat{e}_x$, $\bar{v}_1(0) = 0$, $\bar{v}_2(0) = 0$

Finna staðsetn, hraða og hröðun CM

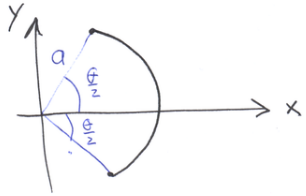
Engin kraftur á $m_2 \to \bar{r}_2(t) = 0$, $\bar{v}_2(0) = 0$

$$\bar{F}_2 = F_0 \hat{e}_x \to m\ddot{\bar{r}}_2 = F_0 \hat{e}_x \to \bar{r}_2(t) = \frac{F_0}{2m} t^2 \cdot \hat{e}_x$$

$$\bar{r}_{cm}(t) = \frac{m_1 \bar{r}_1(t) + m_2 \bar{r}_2(t)}{m_1 + m_2} = \frac{F_0}{4m} t^2 \cdot \hat{e}_x$$

$$\to \bar{v}_{cm} = \frac{F_0}{2m} t \cdot \hat{e}_x \to \bar{a}_{cm} = \frac{F_0}{2m} \hat{e}_x$$

(5) Dæmi 9-04 í bók



Finna CM

samhverfa gefur $Y_{cm} = 0$

$$X_{cm} = \frac{1}{M} \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} x \, dm \qquad M = s\rho = (a\theta) \cdot \rho$$

$$dm = a\rho \, d\theta$$

$$= \frac{1}{M} \int_{-\theta/2}^{\theta/2} d\theta' \cdot a\rho \cdot x \qquad x = a\cos\theta$$

$$= \frac{1}{M} \int_{-\theta/2}^{\theta/2} d\theta' \cdot a^2\rho \cos\theta' = \frac{a^2\rho}{a\theta\rho} \int_{-\theta/2}^{\theta/2} dx \cdot \cos x$$

$$= \frac{a}{\theta} \left\{ \sin x \Big|_{-\theta/2}^{\theta/2} \right\} = \frac{a}{\theta} 2\sin\left(\frac{\theta}{2}\right) = X_{cm}$$

(6) Dæmi 9-20 í bók $\qquad$ reipi á nagla, lengd $2a$

a−x

a+x

$$F_g = \{(a+x)\rho\} g - \{(a-x)\rho\} g$$

$$= (2x\rho) g$$

Finna hraðann þegar reipið fer af naglanum

Hér er m fasti $\qquad \frac{dp}{dt} = F_g$

$$(2a\rho)\dot{v} = (2x\rho) g \to a\dot{v} = xg$$

Vil fyrir hraðann sem fall af x en ekki t

$$\to \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \to a v \frac{dv}{dx} = xg$$

$$\to v \, dv = \frac{xg}{a} dx$$

← på er $a+x = 2a$ og reipið fer fram af

(9)

$$\int_0^v v' dv' = \frac{g}{a}\int_0^a x\,dx \;\rightarrow\; \frac{v^2}{2} = \frac{g}{a}\frac{a^2}{2} = \frac{g}{2}a$$

$$\rightarrow\; v = \sqrt{ga}$$

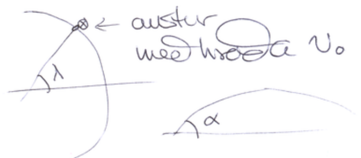
---

② Dæmi 10-09 í bók

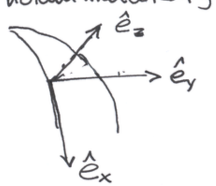
①

Notum $\bar F_{eff} = \bar S + m\bar g - 2m\bar\omega\times\bar v_r$, $\bar S = 0$

$\to \bar a_r = \bar g - 2\bar\omega\times\bar v_r$



notum hnitakerfi eins og í Ex. 10.3



$$\bar a_r = -g\hat a_z - 2(-\omega\cos\lambda,\ 0,\ \omega\sin\lambda) \times (0,\ v_y,\ v_z)$$

$$= -g\hat a_z - 2(-v_y\omega\sin\lambda,\ v_z\omega\cos\lambda,\ -v_y\omega\cos\lambda)$$

finna þver geigum, þ.e. í $\hat e_x$-stefnu

→ skoðum því a_x

$$a_x = 2v_y\omega\sin\lambda = 2v_0\cos\alpha\cdot\omega\cdot\sin\lambda = 2v_0\omega\cos\alpha\sin\lambda$$

②

setjum upphafsgildi

$\dot x(0) = 0$

$x(0) = 0$

$a_x = 2v_0\omega\cos\alpha\sin\lambda$

$\to v_x(t) = 2v_0\omega t\cos\alpha\sin\lambda$

$x(t) = v_0\omega t^2\cos\alpha\cdot\sin\lambda$

Sem fyrstu nálgun gerum við ráð fyrir að z-hreyfingin sé óháð af krafti Coriolis

> Hröðunarþáttur á síðunni að undan sýnir að þetta er vissulega nálgun

$\to z(t) = v_0 t\sin\alpha - \frac{gt^2}{2}$

flugtíminn er því

$$T = \frac{2v_0\sin\alpha}{g}$$

$$x(T) = v_0\omega\left[\frac{2v_0\sin\alpha}{g}\right]^2\cos\alpha\sin\lambda = \frac{4\omega_0 v_0^3}{g^2}\sin\lambda\cos\alpha\sin^2\alpha$$

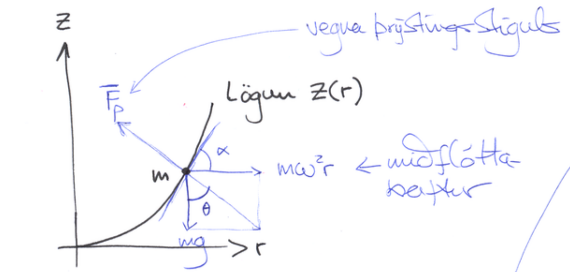
$$\to d = x(T) - x(0) = \frac{4\omega_0 v_0^3}{g^2}\sin\lambda\cos\alpha\sin^2\alpha$$

athyglisvert að v_0 kemur í 3. veldi!

③ Dæmi 10-06 í bók

③

Vatnsfata snýst um snúningsás. Hver er lögun vatnsyfirborðsins?



fyrir m gildir

$$\bar F_{eff} = \bar F - m\ddot{\bar R}_f - m\dot{\bar\omega}\times r - m\bar\omega\times(\bar\omega\times\bar r) - 2m\bar\omega\times\bar v_r$$

Snúningskerfið er með miðju í snúningsás

m hreyfist ekki í því kerfi

$\to \bar F_{eff} = 0$, $\bar v_r = 0$, $\ddot{\bar R}_f = 0$, $\dot{\bar\omega} = 0$

$$0 = \bar F - m\bar\omega\times(\bar\omega\times\bar r)$$

$$\bar F = m\bar g + \bar F_p$$

$\bar F_p$ er hornrétt á yfirborðið ($\bar F_{eff} = 0$), → $\overrightarrow{(\text{þyngd})} + \overrightarrow{(\text{miðflóttakr})}$ eru líka hornrétt á það

því fæst $\alpha = \theta$

$$\tan\theta = \frac{m\omega^2 r}{mg}$$

og $\frac{dz}{dr} = \tan\theta$

(sjá snertil á mynd)

$$\to \frac{dz}{dr} = \frac{\omega^2 r}{g}$$

$$dz = \frac{\omega^2 r}{g}dr$$

$$\to z = \frac{\omega^2}{2g}r^2 + C$$

← heildunarfasti

fleygboga lögun

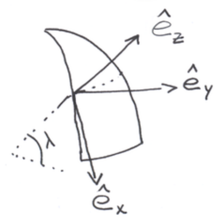
④ Dæmi 10-08 í bók

④

Norðurhvel, ögn hent upp í hæð h yfir punkti á yfirborði jarðar. Sýna hvar hún lendir (hnikast til vesturs)

sleppa loftmótstöðu, $\frac{h}{a_y} \ll 1$

Hnit eins og á mynd 10-9 í bók



Hröðun vegna krafts Coriolis er

$$\bar a = -2\bar\omega\times\dot{\bar r}$$

$$\bar\omega \times \dot{\bar r} = \begin{vmatrix} \bar i & \bar j & \bar k \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & \dot z \end{vmatrix} = (0,\ \omega\cos\lambda\cdot\dot z,\ 0)$$

($\dot z$ ←) því nálgun!

$$\rightarrow \bar a = -2\bar\omega\times\dot{\bar r} = -2\omega\dot z\cos\lambda\,\hat e_y$$

$$\rightarrow \ddot y = -2\omega\dot z\cos\lambda$$

Fyrir z-stefnu

$$z = v_0 t - \frac{gt^2}{2}$$

$$v_z^2 = v_0^2 - 2zg$$

$$\rightarrow \quad v_0 = \sqrt{2gh}$$

heildum í tíma

$$\dot y = -2\omega z\cos\lambda + C_1$$

$$\dot y(0) = 0 \rightarrow C_1 = 0$$

$$\dot y = -2\omega\left\{v_0 t - \frac{gt^2}{2}\right\}\cos\lambda$$

$$= -\omega\cos\lambda\cdot\left\{2v_0 t - gt^2\right\}$$

heildum aftur

$$y = -\omega\cos\lambda\cdot\left\{v_0 t^2 - \frac{gt^3}{3}\right\} + C_2$$

$$y(0) = 0 \rightarrow C_2 = 0$$

$$y = -\omega\cos\lambda\cdot\left\{v_0 t^2 - \frac{gt^3}{3}\right\}$$

flugtími: $T = \frac{2v_0}{g}$

$$\rightarrow y = -\omega\cos\lambda\cdot\left\{v_0\left(\frac{2v_0}{g}\right)^2 - \frac{g}{3}\left(\frac{2v_0}{g}\right)^3\right\}$$

$$= -\omega\cos\lambda\cdot\frac{v_0^3}{g^2}\cdot\left\{4 - \frac{8}{3}\right\}$$

$$= -\omega\cos\lambda\cdot\frac{4}{3}\frac{v_0^3}{g^2}$$

$$v_0 = \sqrt{2gh}$$

$$\rightarrow y = -\omega\cos\lambda\cdot\frac{4}{3g^2}(2gh)^{3/2}$$

$$= -\frac{4\omega}{3}\cos\lambda\cdot\sqrt{\frac{8h^3}{g}}$$

↑ í vestur átt

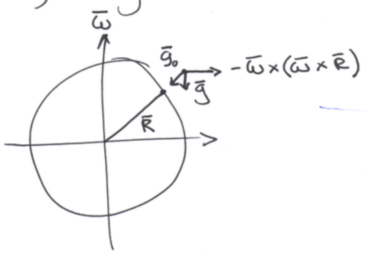
(5) Dæmi 10-17 í bók

Stöðuvatn nálgað sem hringur („spherical cow")

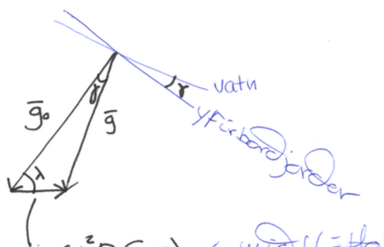
geisli 162 km, $\lambda = 47^\circ$

Finna hve mikið miðsóknarkrafturinn lækkar miðjuna m.v. strönd

Sjá mynd 10-6 í bók



teiknum vigrana aftur



$\omega^2 R\cos\lambda$ ← miðflóttahröðun

$$|g| = \left\{(\omega^2R\cos\lambda)^2 + (g_0)^2 - 2(\omega^2R\cos\lambda)\,g_0\cos\lambda\right\}^{1/2}$$

↑ cosínusregla

$$\frac{|g|}{\sin\lambda} = \frac{(\omega^2R\cos\lambda)}{\sin\gamma}$$

Sínus-regla

$$\rightarrow \gamma = \arcsin\left\{\frac{(\omega^2R\cos\lambda)\sin\lambda}{|g|}\right\}$$

$$h = r\sin\gamma$$

$$\rightarrow h = r\left\{\frac{\omega^2R\cos\lambda\cdot\sin\lambda}{\sqrt{(\omega^2R\cos\lambda)^2 + g_0^2 - 2g_0\omega^2R\cos\lambda}}\right\} \sim \frac{r\omega^2R\cos\lambda\sin\lambda}{g_0}$$

↑ geisli vatns

$r = 162$ km

$\omega = 7.3\cdot10^{-5}$ rad/s

$\lambda = 47^\circ \sim 0{,}82$ rad

$R = 6.4\cdot10^{6}$ m

$g_0 = 9.81$ m/s²

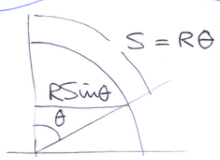
Ég fæ $h \sim 7{,}1$ m

(6) Dæmi 10-11 í bók   Loftmótstöðu sleppt   (9)

Ögn send frá N-skauti (norðri yfirb. jarðar) hvar gengur?

$T = 10\,\text{min}$, $S = 4800\,\text{km}$

Byrjar á N-skauti, sem er fast í fasta hnitakerfinu → notum það. Jörðin snýst meðan á því stendur

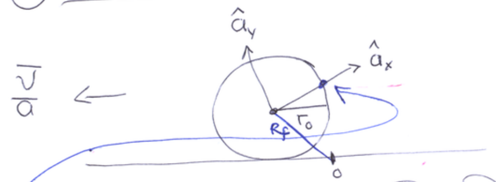


$$\Delta L = \omega_y R \sin\theta \cdot T = \omega_y R \sin\left(\frac{S}{R}\right)\frac{S}{v}$$

$$\Delta\phi = \left(\frac{\Delta L}{2\pi R\sin\theta}\right)2\pi = \omega_y \cdot T = \omega_y \cdot \frac{S}{v}$$

$\Delta L \approx 191$ km

(1) Dæmi 10-02 í bók   (10)



Finna punktinn á dekkinu með mesta hröðun m.v. jörð, hver er hröðunin?

Snúningshnitakerfi með miðju í miðju hjóls, þar er hjólið kyrrt. Sleppum þyngdarhröðun. Fyrir punkt á dekkinu m.v. jörð gildir (10.23)

$$\bar{a}_f = \ddot{\bar{R}}_f + \bar{a}_r + \dot{\bar{\omega}}\times\bar{r} + \bar{\omega}\times(\bar{\omega}\times\bar{r}) + 2\bar{\omega}\times\bar{v}_r$$

Í snúningshnitakerfinu notum við

$$\ddot{\bar{R}}_f = -a\cos\theta\cdot\hat{e}_x + a\sin\theta\,\hat{e}_y$$

$\bar{r} = r_0\hat{e}_x$ staðsetning punkts á dekki

$\bar{v}_r = 0 \qquad \bar{\omega} = \frac{v}{r_0}\hat{e}_z$

$\bar{a}_r = 0 \qquad \dot{\bar{\omega}} = \frac{a}{r_0}\hat{e}_z$

$$2\bar{\omega}\times\bar{v}_r = 0$$

$$\bar{\omega}\times(\bar{\omega}\times\bar{r}) = \left(\frac{v}{r_0}\right)^2 r_0 \cdot \hat{e}_z\times(\hat{e}_z\times\hat{e}_x) = \frac{v^2}{r_0}\hat{e}_z\times\hat{e}_y = -\frac{v^2}{r_0}\hat{e}_x$$

$$\dot{\bar{\omega}}\times\bar{r} = a\,\hat{e}_z\times\hat{e}_x = a\hat{e}_y$$

(11)

$$\to \bar{a}_f = -a\cos\theta\cdot\hat{e}_x + a\sin\theta\cdot\hat{e}_y + a\hat{e}_y - \frac{v^2}{r_0}\hat{e}_x$$

$$= \hat{e}_x\left\{-a\cos\theta - \frac{v^2}{r_0}\right\} + \hat{e}_y\left\{a\sin\theta + a\right\}$$

$$\to \bar{a}_f\cdot\bar{a}_f = |a_f|^2 = \left(a\cos\theta + \frac{v^2}{r_0}\right)^2 + a^2(\sin\theta+1)^2$$

$$= a^2\cos^2\theta + 2a\cos\theta\cdot\frac{v^2}{r_0} + \left(\frac{v^2}{r_0}\right)^2 + a^2\sin^2\theta + a^2 + a^2 2\sin\theta$$

$$= \frac{v^4}{r_0^2} + 2a^2 + \frac{2av^2}{r_0}\cos\theta + a^2 2\sin\theta$$

finnum útgildi

$$\frac{d|a_f|^2}{d\theta} = -\frac{2av^2}{r_0}\sin\theta + a^2 2\cos\theta = 0$$

$$\text{ef}\quad \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{ar_0}{v^2}$$

(12)

$$\tan\theta = \frac{ar_0}{v^2}$$

$$\to \cos\theta = \frac{v^2}{\sqrt{(ar_0)^2+v^4}}, \quad \sin\theta = \frac{ar_0}{\sqrt{(ar_0)^2+v^4}}$$

{og eðlilega $\cos^2\theta + \sin^2\theta = 1$}

$$\to \bar{a}_f = -\hat{e}_x\left\{\frac{av^2}{\sqrt{(ar_0)^2+v^4}} + \frac{v^2}{r_0}\right\} + \hat{e}_y a\left\{\frac{ar_0}{\sqrt{(ar_0)^2+v^4}} + 1\right\}$$

$$\to |a_f|^2 = \frac{v^4}{r_0^2} + 2a^2 + \frac{2av^2}{r_0}\frac{v^2}{\sqrt{(ar_0)^2+v^4}} + a^2 2\frac{ar_0}{\sqrt{(ar_0)^2+v^4}}$$

$$= \frac{v^4}{r_0^2} + 2a^2 + \frac{\frac{2av^4}{r_0} + \frac{2a}{r_0}(ar_0)^2}{\sqrt{(ar_0)^2+v^4}}$$

$$\rightarrow |a_f|^2 = \frac{v^4}{r_0^2} + 2a^2 + \frac{2a}{r_0} \frac{v^4 + (ar_0)^2}{\sqrt{(ar_0)^2 + v^4}}$$

$$= \frac{v^4}{r_0^2} + 2a^2 + \frac{2a}{r_0} \sqrt{v^4 + (ar_0)^2}$$

$$= \frac{v^4}{r_0^2} + 2a^2 + 2a\sqrt{\frac{v^4}{r_0^2} + a^2}$$

$$= \left(a + \sqrt{\frac{v^4}{r_0^2} + a^2}\right)^2$$

$$\rightarrow |a_f| = a + \sqrt{\frac{v^4}{r_0^2} + a}$$

① Dæmi 11-01 í bók

Finna I_1, I_2, og I_3 fyrir kúlu með M og R, gegnheil kúla.
Notum kúluhnit með miðju í miðju kúlu

$$I_{ij} = \int_V dv\, \rho(\vec{r})\left\{\delta_{ij}\sum_k x_k^2 - x_i x_j\right\}$$

Vegna samhverfu og kúluhnita er heppilegt að reikna

$$I_{33} = \rho\int_V dv\,(r^2 - z^2) = \rho\int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos\theta)\int_0^R r^2 dr\cdot r^2(1-\cos^2\theta)$$

$$= 2\pi\rho\int_0^R dr\, r^4\int_{-1}^{+1} d(\cos\theta)(1-\cos^2\theta) = 2\pi\rho\frac{R^5}{5}\cdot\frac{4}{3}$$

$$M = \frac{4\pi}{3}\rho R^3 \quad \to \quad I_{33} = \frac{2}{5}MR^2$$

Kúlan er kúlusamhverf um miðpunkt $\to I_{11} = I_{22} = I_{33}$

Athugum I_{ij} utan hornalínu

$$I_{ij} = \rho\int dv\,\{-x_i x_j\} \quad \text{ef } i \neq j$$

Öll þessi heildi hverfa eins og sést í kartesískum hnitum með miðju í kúlu vegna andsamhverfu um 0

Því er hverfitregðuþinurinn á hornalínu hans

$$\mathbb{I} = \begin{Bmatrix} I_{33} & 0 & 0 \\ 0 & I_{33} & 0 \\ 0 & 0 & I_{33} \end{Bmatrix}$$

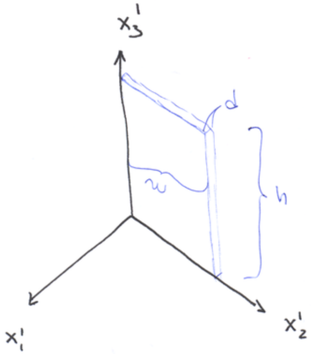
og því eru höfuðásarnir þrír hornréttir ásar um 0 og hverfitregður um þá eru $I_1 = I_2 = I_3 = \frac{2}{5}MR^2$

② Dæmi 11-08 í bók

Hurð með breidd $w = 1m$

Ef hún er opnuð um 90° fellur hún að stöfum á 2s.

Sýnið að lína um hjarir hlýtur að halla 3° frá lóðréttu.



$w = 1m$

$M = \rho w h d$

$\rho = \frac{M}{whd}$

$d \ll w, h$

$$I_3 = \rho d\int_0^h dh'\int_0^w dw'\{h'^2 + w'^2 - h'^2\}$$

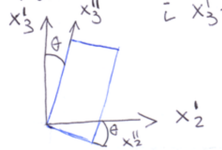
$$= \rho d\int_0^h dh'\int_0^w dw'\cdot w'^2$$

$$= \frac{M}{whd}hd\cdot\frac{w^3}{3} = \frac{1}{3}Mw^2$$

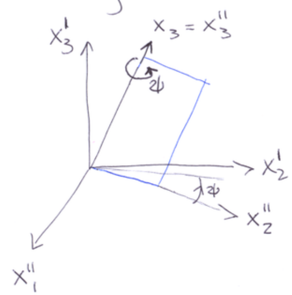
Stillum hurð með halla í upphafi, horn Eulers

Engin snúningur um x_3'
$\to \phi = 0$

Höllum $\to$ snúningur um $-\theta$ í x_3'-x_2'-sléttu



Snúningur um ψ um x_3''-ás $\to x_3$ (11.99)



$$\left.\begin{matrix} \phi = 0 \\ -\theta \\ \psi \end{matrix}\right\} \to \lambda = \begin{pmatrix} \cos\psi & \cos\theta\sin\psi & -\sin\psi\sin\theta \\ -\sin\psi & \cos\theta\cos\psi & -\cos\psi\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Hurðin fellur að stöfum með $-\psi$ snúningi

$$\omega_1 = 0,\ \omega_2 = 0$$

$$\to I_3\dot{\omega}_3 = I_3\ddot{\psi} = N_3$$

Í hnitakerfi hurðar er CM:

$$\bar{R} = \begin{pmatrix} 0 \\ w/2 \\ h/2 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

ef $d \simeq 0$

Í kerfi dyrastafanna er

$$\bar{R}' = \lambda^{-1}\bar{R} = \lambda^t\bar{R} = \frac{1}{2}\begin{pmatrix} -w\sin\psi \\ w\cos\theta\cos\psi + h\sin\theta \\ -w\cos\psi\sin\theta + h\cos\theta \end{pmatrix}$$

(5)

Kraftar þyngdar á hurðina í kerfi Stafa

$$\bar{F}' = -Mg\hat{e}_3'$$

$$\rightarrow \bar{N}' = \bar{R}' \times \bar{F}' = -\frac{M}{2}g \begin{vmatrix} \hat{e}_1' & \hat{e}_2' & \hat{e}_3' \\ -w\sin\psi, & w\cos\psi\cos\theta + h\sin\theta, & -w\cos\psi\sin\theta + h\cos\theta \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -\frac{Mg}{2}\begin{pmatrix} w\cos\psi\cos\theta + h\sin\theta \\ w\sin\psi \\ 0 \end{pmatrix}$$

Í kerfi hurðar fæst því

$$\bar{N} = \lambda\bar{N}' = -\frac{Mg}{2}\begin{pmatrix} w\cos^2\psi\cos\theta + h\sin\theta\cos\psi + w\sin^2\psi\cos\theta \\ -h\sin\theta\sin\psi \\ w\sin\theta\sin\psi \end{pmatrix}$$

(6)

$$\rightarrow N_3 = -\frac{Mg}{2}w\sin\theta\sin\psi$$

og með hreyfijöfnunni

$$I_3\ddot{\psi} = N_3 = -\frac{Mg}{2}w\sin\theta\sin\psi$$

$$I_3 = \frac{1}{3}Mw^2$$

$$\rightarrow \boxed{\ddot{\psi} = -\frac{3}{2}\frac{g}{w}\sin\theta\sin\psi}$$ sem við þurfum að leysa

margföldum með $\dot{\psi}$

$$\dot{\psi}\ddot{\psi} = \left(-\frac{3}{2}\frac{g}{w}\sin\theta\right)\dot{\psi}\sin\psi$$

heildum óákveðið m.t.t. t

$$\frac{1}{2}\dot{\psi}^2 = \left(\frac{3g}{2w}\sin\theta\right)\cos\psi + C_1$$

Opnuðum 90°

$\cos(\psi(0)) = 0$

$\dot{\psi}(0) = 0$

$\rightarrow C_1 = 0$

(7)

$$\rightarrow \dot{\psi} = \pm\sqrt{\frac{3g}{w}\sin\theta\cos\psi}$$

hurðin fellur að stöfum að $\dot{\psi} < 0$ þegar $\cos\psi > 0$

$$\int_{\pi/2}^{0}\frac{d\psi}{\sqrt{\cos\psi}} = -\sqrt{\frac{3g}{w}\sin\theta}\int_0^T dt$$

(T:) tíminn sem var gefinn sem 2s

$$= -T\sqrt{\frac{3g}{w}\sin\theta}$$

$$\rightarrow \int_0^{\pi/2}\frac{d\psi}{\sqrt{\cos\psi}} = T\sqrt{\frac{3g}{w}\sin\theta}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$ (GR: 8.384.1)

$$B(\tfrac{1}{4},\tfrac{1}{4}) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})}$$

$$\int_0^{\pi/2}\cos^{\mu-1}x\,dx = 2^{\mu-2}B\left(\frac{\mu}{2},\frac{\mu}{2}\right)$$

(GR: 3.621.1)

hér $\mu = \frac{1}{2}$

$$\rightarrow \int_0^{\pi/2}\cos^{-1/2}x\,dx = 2^{-3/2}B(\tfrac{1}{4},\tfrac{1}{4})$$

(8)

$$\rightarrow \int_0^{\pi/2}\cos^{-1/2}x\,dx = \frac{(\Gamma(\frac{1}{4}))^2}{\Gamma(\frac{1}{2})}\left(\frac{1}{\sqrt{2}}\right)^3 \approx 2.6221 \approx T\sqrt{\frac{3g}{w}\sin\theta}$$

$$\rightarrow \sin\theta \approx \frac{w}{3gT^2}\cdot(2.621)^2$$

$g = 9.81\ m/s^2$

$$\theta \approx \arcsin\left(\frac{w}{3gT^2}(2.621)^2\right) \simeq 0{,}0584\ \text{rad} \simeq 3.35°$$

(3) Dæmi 11-06 í bók

Hvernig getum við greint á milli?

Ein kúla gegnheil

Önnur kúluskel

Sami M og R

Finnum hverfitregðuna um miðju þeirra

Hér að undan sáum við að fyrir gegnheila kúlu

fæst $I_s = \frac{2}{5}MR^2$ fyrir ás í gegnum miðju.

Kúluskel

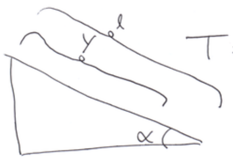
Gefum okkur að hún sé þunn $\rightarrow$ $M = 4\pi R^2 \cdot \rho_s$ ← Flatar þéttleiki massans

Reiknum fyrir $\hat{e}_2$-ás um miðju (það er höfuðás)

$$I_h = \rho_s \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \left\{R^2 - z^2\right\} R^2 \qquad z = R\cos\theta$$

$$= \rho_s \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin^3\theta \cdot R^4 = \frac{8}{3}\pi\rho_s R^4 = \frac{2}{3}MR^2$$

Ef við látum kúlurnar rúlla niður skáplan má bera við mismuni.



$$T = \frac{M}{2}\dot{y}^2 + \frac{I}{2}\dot{\theta}^2, \qquad U = Mg(l-y)\sin\alpha, \qquad y = R\theta$$

$$L = \frac{M}{2}\dot{y}^2 + \frac{I}{2R^2}\dot{y}^2 + Mgy\sin\alpha$$

Sleppum

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \qquad L = \left\{\frac{M}{2} + \frac{I}{2R^2}\right\}\dot{y}^2 + Mgy\sin\alpha$$

$$\frac{\partial L}{\partial y} = Mg\sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}\left(\dot{y}\left\{M + \frac{I}{R^2}\right\}\right) = \ddot{y}\left\{M + \frac{I}{R^2}\right\}$$

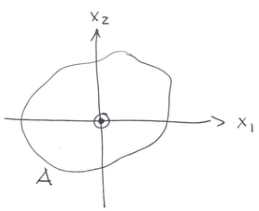
$$\rightarrow \quad Mg\sin\alpha - \ddot{y}\left\{M + \frac{I}{R^2}\right\} = 0 \quad \rightarrow \quad \ddot{y} = \frac{gMR^2\sin\alpha}{\{MR^2 + I\}}$$

$\rightarrow$ Kúlan með minni I hraðast meir niður skáplanið

$$I_h = \frac{2}{3}MR^2 \qquad I_s = \frac{2}{5}MR^2$$

$\frac{2}{5} < \frac{2}{3}$ $\rightarrow$ gegnheila kúlan fer hraðar

(4) Dæmi 11-17 í bók



Miðum við O

Einsleit plata í x_1-x_2-sléttu

$$I_{11} = \rho_s \int_A dx_1 dx_2 \left\{r^2 - x_1^2\right\}$$

$$= \rho_s \int_A dx_1 dx_2 \left\{x_2^2 + x_3^2\right\} = \rho_s \int dx_1 dx_2 \cdot x_2^2 \equiv A$$

því $x_3 = 0$ í sléttunni

$$I_{22} = \rho_s \int dx_1 dx_2 \left\{r^2 - x_2^2\right\}$$

$$= \rho_s \int dx_1 dx_2 \left\{x_1^2 + x_3^2\right\} = \rho \int dx_1 dx_2 \cdot x_1^2 \equiv B$$

$$I_{33} = \rho_s \int dx_1 dx_2 \left\{r^2 - x_3^2\right\} = \rho \int dx_1 dx_2 \left\{x_1^2 + x_2^2\right\} = A + B$$

$$I_{13} = I_{31} = \rho_s \int dx_1 dx_2 \left\{-x_1 x_3\right\} = 0$$

$$I_{23} = I_{32} = \rho_s \int dx_1 dx_2 \left\{-x_2 x_3\right\} = 0$$

$$I_{12} = I_{21} = \rho_s \int dx_1 dx_2 \left\{-x_1 x_2\right\} \equiv -C$$

$$\Rightarrow \mathbb{I} = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

Hér má því finna almenna jöfnu sem aðeins inniheldur A, B, og C fyrir höfuðásunum þó lögunin sé ekki alveg gefin.

(5) Dæmi 11-18 í bók

Ef í dæminu að undan ásarnum er snúið um θ um x_3-ás finna $\mathbb{I}'$

Aðeins snúningur um x_3'- ás – (11.91) $\rightarrow \lambda = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\mathbb{I}' = \lambda \mathbb{I} \lambda^t$$

$$= \begin{bmatrix} A\cos^2\theta - C\sin(2\theta) + B\sin^2\theta & -C\cos(2\theta) + \frac{B-A}{2}\sin(2\theta) & 0 \\ -C\cos(2\theta) + \frac{B-A}{2}\sin(2\theta) & A\sin^2\theta + C\sin(2\theta) + B\cos^2\theta & 0 \\ 0 & 0 & A+B \end{bmatrix}$$

$$= \begin{bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A+B \end{bmatrix}$$

með

$$A' = A\cos^2\theta - C\sin(2\theta) + B\sin^2\theta$$
$$B' = A\sin^2\theta + C\sin(2\theta) + B\cos^2\theta$$
$$C' = C\cos(2\theta) - \frac{B-A}{2}\sin(2\theta)$$

fyrir x_1 og x_2 að verða höfuðása þarf að gilda

$$C' = C\cos(2\theta) - \frac{B-A}{2}\sin(2\theta) = 0$$

$$\rightarrow \frac{\sin(2\theta)}{\cos(2\theta)} = \tan(2\theta) = \left(\frac{2C}{B-A}\right)$$

$$\rightarrow \theta = \frac{1}{2}\arctan\left(\frac{2C}{B-A}\right)$$

(6) Dæmi 11-16 í bók

$$\mathbb{I} = \begin{bmatrix} \frac{A+B}{2} & \frac{A-B}{2} & 0 \\ \frac{A-B}{2} & \frac{A+B}{2} & 0 \\ 0 & 0 & C \end{bmatrix}$$

Snúa um θ um x_3- ás

$$\lambda = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{I}' = \lambda \mathbb{I} \lambda^t$$

$$= \begin{bmatrix} \frac{A+B}{2} + (A-B)\cos\theta\sin\theta & \frac{A-B}{2}\cos^2\theta - \frac{A-B}{2}\sin^2\theta & 0 \\ -\frac{A-B}{2}\sin^2\theta + \frac{A-B}{2}\cos^2\theta & \frac{A+B}{2} - \frac{A-B}{2}\cos\theta\sin\theta & 0 \\ 0 & 0 & C \end{bmatrix}$$

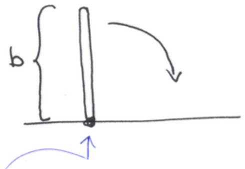
ef $\theta = \frac{\pi}{4}$

$\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

þá fæst $\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$

① Dæmi 11-20 í bók

einsleit stöng með lengd fellur
finna hornferðina þ. hún skellur á gólfið.



Einfaldast að nota orkuvarðveislu

Í upphafi $E_1 = U = mg\frac{b}{2}$ ← m.v. CM

Í lok $E_2 = T = \frac{I}{2}\omega^2$

Um endapunkt $I = I_{33} = \int_0^b dx\, \rho_l \left(x^2 - z^2\right) = \rho_l \int_0^b dx \cdot x^2$

$$= \rho_l \frac{1}{3} b^3 = \frac{1}{3}(\rho_l b) b^2 = \frac{1}{3} m b^2$$

$$\to E_2 = \frac{1}{6} m b^2 \omega^2 = E_1 = mg\frac{b}{2} \to \omega = \sqrt{\frac{3g}{b}}$$

② Dæmi 11-22 og -23 í bók

$$tr\{\mathbb{I}\} = \sum_k I_{kk}$$

Sýnið að einslögunar ummyndun breyti ekki spori þess

Einóta ummyndun

$\mathbb{I}' = \lambda \mathbb{I} \lambda^t$ eins gildir $\lambda\lambda^t = \mathbb{1}$

$$tr\{\mathbb{I}'\} = tr\{\lambda \mathbb{I} \lambda^t\} = tr\{\lambda^t \lambda \mathbb{I}\} = tr\{\mathbb{I}\}$$

Sýnið að samskonar gildi fyrir ákveðu

$$\det\{\lambda \mathbb{I} \lambda^t\} = \det\{\lambda\}\det\{\mathbb{I}\}\det\{\lambda^t\}$$

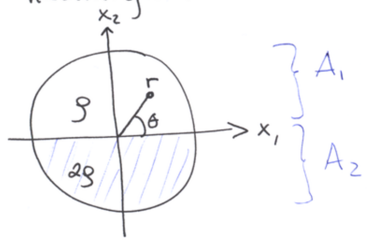
$$\det\{\lambda\lambda^t\} = \det\{\mathbb{1}\} = 1$$

$$\to \det\{\lambda \mathbb{I} \lambda^t\} = \det\{\mathbb{I}\}$$

Skoðið endilega
„The matrix Cookbook"
á veraldarvefnum

③ Dæmi 11-25 í bók

Þunnur diskur settur saman úr tveimur einsleitum helmingum



finnum fall Lagrange fyrir veltu disksins.

Við þurfum að finna CM og I í gegnum CM

Miðað við teikninguna hér að framan er CM fyrir neðan O-hnitapunktinn á x_2-ás

$$x_2^{CM} = \frac{\rho}{M}\left\{2\int_{A_2} x_2 dA + \int_{A_1} x_2 dA\right\}$$

$$= \frac{\rho}{M}\left\{2\int_0^R r dr \int_\pi^{2\pi} d\theta\, r\sin\theta + \int_0^R r dr \int_0^\pi d\theta\, r\sin\theta\right\}$$

$$= \frac{\rho}{M}\frac{R^3}{3}\left\{-2\cos\theta\Big|_\pi^{2\pi} - \cos\theta\Big|_0^\pi\right\} = \frac{\rho R^3}{3M}\left\{-2(1+1)-(-1-1)\right\}$$

$$= -\frac{2\rho R^3}{3M}$$

Massi disksins er

$$M = \frac{2\cdot\pi R^2\rho}{2} + \frac{\pi R^2\rho}{2} = \frac{3}{2}\rho\pi R^2$$

$$\to \quad x_2^{CM} = -\frac{2\rho R^3}{3}\,\frac{2}{3\rho\pi R^2} = -\frac{4R}{9\pi}$$

Næst reiknum við I m.v. O, þ.e. miðju hjóls

$$I_{33} = \rho\left\{2\int_0^R (r^2 - z^2)\, r\,dr \int_\pi^{2\pi} d\theta + \int_0^R (r^2 - z^2)\, r\,dr \int_0^\pi d\theta\right\}$$

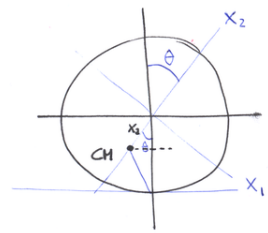
$z = 0$

$$= \rho\frac{R^4}{4}\left\{2\pi + \pi\right\} = \frac{3}{4}\rho\pi R^4 = \frac{1}{2}MR^2 = I_3$$

Notum setningu Steiners til að reikna I_3 í gegnum CM

$$I_0 = I_3 - M(x_2^{CM})^2 = \frac{1}{2}MR^2 - M\frac{16R^2}{81\pi^2} = \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right]$$

Þurfum hraða CM



Í föstu hnitakerfi breytur

$$x_{CM} = R\theta - |x_2^{CM}|\sin\theta$$
$$y_{CM} = R - |x_2^{CM}|\cos\theta$$

$$\dot{x}_{CM} = R\dot{\theta} - |x_2^{CM}|\dot{\theta}\cos\theta$$
$$\dot{y}_{CM} = |x_2^{CM}|\dot{\theta}\sin\theta$$

$$V^2 = (\dot{x}_{CM}^2 + \dot{y}_{CM}^2) = \left\{R\dot{\theta} - |x_2^{CM}|\dot{\theta}\cos\theta\right\}^2 + \left\{|x_2^{CM}|\dot{\theta}\sin\theta\right\}^2$$

$$= (R\dot{\theta})^2 + (|x_2^{CM}|\dot{\theta})^2 - 2\dot{\theta}^2 R|x_2^{CM}|\cos\theta$$

$$= \dot{\theta}^2\left\{R^2 + |x_2^{CM}|^2 - 2R|x_2^{CM}|\cos\theta\right\}$$

$$V^2 = \dot{\theta}^2 R^2\left\{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta\right\}$$

$$T = \frac{1}{2}MV^2 + \frac{I_0}{2}\dot{\theta}^2 = \frac{1}{2}M(R\dot{\theta})^2\left\{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta\right\}$$
$$+ \frac{1}{2}M(R\dot{\theta})^2\left\{\frac{1}{2} - \frac{16}{81\pi^2}\right\}$$
$$= \frac{1}{2}M(R\dot{\theta})^2\left\{\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right\}$$

$$U = Mg\,y_{CM}, \quad y_{CM} = R - |x_2^{CM}|\cos\theta = R\left\{1 - \frac{4}{9\pi}\cos\theta\right\}$$

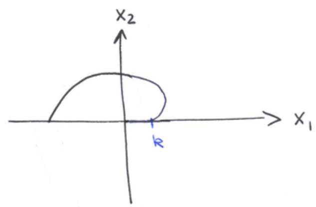
Við getum notað $U = Mg\,y_{CM} - Mg\frac{R}{2} = \frac{1}{2}Mg\left\{\frac{R}{2} - \frac{8}{9\pi}\cos\theta\right\}$

$$\to L = \frac{1}{2}M(R\dot{\theta})^2\left\{\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right\} - \frac{1}{2}Mg\left\{\frac{R}{2} - \frac{8}{9\pi}\cos\theta\right\}$$

---

(4) Dæmi 11-19 í bók

Plata (einsleit) með ρ takmörkuð af $r = ke^{\alpha\theta}$ og $\theta = 0$ og $\theta = \pi$. Finnum hverfitregðu þinina um O $(r=0)$ ef platan er í x_1-x_2-sléttunni



notum pól hnit

$$x_1 = \eta\cos\theta$$
$$x_2 = \eta\sin\theta$$

(9)

$$I_{11} = \rho\int_0^{\pi} d\theta \int_0^{ke^{\alpha\theta}} (x_1^2 + x_2^2 + x_3^2 - x_1^2)\,\eta\, d\eta = \rho\int_0^{\pi} d\theta \int_0^{ke^{\alpha\theta}} x_2^2\,\eta\, d\eta$$

($x_3 = 0$)

$$= \rho\int_0^{\pi} d\theta \sin^2\theta \int_0^{ke^{\alpha\theta}} \eta^3 d\eta = \rho\int_0^{\pi} d\theta \sin^2\theta \, \frac{k^4 e^{4\alpha\theta}}{4}$$

$$= \frac{\rho k^4}{4}\int_0^{\pi} d\theta \sin^2\theta\, e^{4\alpha\theta} \overset{(E.18b)}{=} \frac{\rho k^4}{4}\left\{ \frac{e^{4\alpha\theta}}{(4\alpha)^2+4}\left(4\alpha \sin^2\theta - 2\sin\theta\cos\theta + \frac{2}{4\alpha}\right)\Bigg|_0^{\pi} \right\}$$

$$= \frac{\rho k^4}{4}\left\{ \frac{e^{4\alpha\pi}}{(4\alpha)^2+4}\,\frac{2}{4\alpha} - \frac{1}{(4\alpha)^2+4}\cdot\frac{2}{4\alpha} \right\}$$

(10)

$$\rightarrow I_{11} = \frac{\rho k^4}{2\alpha}\left[\frac{e^{4\pi\alpha} - 1}{16(4\alpha^2+1)} \right]$$

$$I_{22} = \rho\int_0^{\pi} d\theta \int_0^{ke^{\alpha\theta}} x_1^2\,\eta\, d\eta = \rho\int_0^{\pi} d\theta \cos^2\theta \int_0^{ke^{\alpha\theta}} \eta^3 d\eta$$

$$= \rho\int_0^{\pi} d\theta (1-\sin^2\theta)\int_0^{ke^{\alpha\theta}} \eta^3 d\eta = \rho\int_0^{\pi} d\theta \frac{(ke^{\alpha\theta})^4}{4} - I_{11}$$

$$= \frac{\rho k^4}{4}\int_0^{\pi} d\theta\, e^{4\alpha\theta} - I_{11} = \frac{\rho k^4}{4}\left\{\frac{e^{4\alpha\pi}-1}{4\alpha}\right\} - I_{11}$$

$$= \frac{\rho k^4}{2\alpha}\left\{\frac{e^{4\alpha\pi}-1}{8}\right\} - I_{11}$$

(11)

$$\rightarrow I_{22} = \frac{\rho k^4}{2\alpha}\left\{\frac{e^{4\pi\alpha}-1}{16(4\alpha^2+1)}\right\}\left\{8\alpha^2+1\right\}$$

$$I_{12} = -\rho\int_0^{\pi} d\theta \int_0^{ke^{\alpha\theta}} \eta\, d\eta\, x_1 x_2 = -\rho\int_0^{\pi} d\theta \cos\theta\sin\theta \int_0^{ke^{\alpha\theta}} \eta^3 d\eta$$

notum $\cos\theta\sin\theta = \frac{\sin(2\theta)}{2}$

$$I_{12} = -\rho\int_0^{\pi} d\theta \frac{\sin(2\theta)}{2}\int_0^{ke^{\alpha\theta}} \eta^3 d\eta = -2\alpha I_{11}$$

$$= -\rho k^4\left\{\frac{e^{4\pi\alpha}-1}{16(4\alpha^2+1)}\right\}$$

notum 11-17

$$\rightarrow I_{33} = I_{11} + I_{22}$$

$$= \frac{\rho k^4}{2\alpha}\left\{\frac{e^{4\pi\alpha}-1}{16(4\alpha^2+1)}\right\}\left\{8\alpha^2+2\right\}$$

(12)

$$\rightarrow \mathbb{I} = \begin{Bmatrix} F & -2\alpha F & 0 \\ -2\alpha F & F(8\alpha^2+1) & 0 \\ 0 & 0 & F(8\alpha^2+2) \end{Bmatrix}$$

$$= F\begin{Bmatrix} 1 & -2\alpha & 0 \\ -2\alpha & (8\alpha^2+1) & 0 \\ 0 & 0 & (8\alpha^2+2) \end{Bmatrix}$$

$I'_3 = I'_1 + I'_2$

Eigingildin og því höfuðása hverfitregður eru:

$$I'_1 = F\left\{(4\alpha^2+1) - 2\alpha\sqrt{4\alpha^2+1}\right\},\quad I'_2 = F\left\{(4\alpha^2+1) + 2\alpha\sqrt{4\alpha^2+1}\right\},\quad I'_3 = F\left\{8\alpha^2+2\right\}$$

(5) Dæmi 11-13 í bók

3-punktmassar

$m_1 = 3m$ í $(b,0,b)$

$m_2 = 4m$ í $(b,b,-b)$

$m_3 = 2m$ í $(-b,b,0)$

finnum $\mathbb{I}$, höfuðása og hverfitregðu um höfuðása

$$I_{11} = \sum_\alpha m_\alpha\left\{x_{\alpha,2}^2 + x_{\alpha,3}^2\right\} = 3m\{b^2\} + 4m\{2b^2\} + 2m\{b^2\} = 13mb^2$$

$I_{22} = 16mb^2$ $\quad$ $I_{12} = -2mb^2$

$I_{33} = 15mb^2$ $\quad$ $I_{13} = mb^2$

$I_{23} = 4mb^2$

$$\mathbb{I} = \begin{Bmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{Bmatrix} mb^2$$

Hverfitregður um höfuðása er

$I_1 = mb^2\{17 - \sqrt{7}\} \simeq 14.354\, mb^2$

$I_2 = mb^2\{17 + \sqrt{7}\} \simeq 19.646\, mb^2$

$I_3 = mb^2\{10\} = 10\, mb^2$

(eigingildi)

með höfuðása (á stöðluðu)

1: $(1, \sqrt{7}-3, \sqrt{7}-2)b$

2: $(1, -\sqrt{7}-3, -\sqrt{7}-2)b$

3: $(1,1,-1)b$

(eiginvigrar)

(6) Dæmi 11-30 í bók

Viðsnúningspunktar fyrir vagg (nutation) í snúð

(11.162)
$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)$$

þegar $\dot{\theta} = 0$

$$\rightarrow E' = V(\theta) = \frac{\{P_\phi - P_\psi \cos\theta\}^2}{2I_1 \sin^2\theta} + Mgh\cos\theta$$

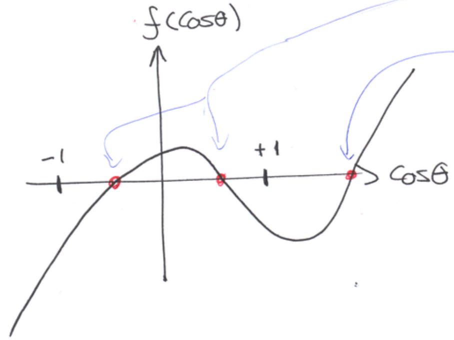
$$\rightarrow \{P_\phi - P_\psi\cos\theta\}^2 + 2I_1 Mgh\sin^2\theta\cos\theta - E' 2I_1\sin^2\theta = 0$$

$$\{P_\phi - P_\psi\cos\theta\}^2 + 2I_1 Mgh(1-\cos^2\theta)\cos\theta - E' 2I_1(1-\cos^2\theta) = 0$$

$$\{2MghI_1\}\cos^3\theta - \{2E'I_1 + P_\psi^2\}\cos^2\theta + 2\{P_\phi P_\psi - MghI_1\}\cos\theta + \{2E'I_1 - P_\phi^2\} = 0$$

3. stigs margliða í $\cos\theta$, $f(\cos\theta) = 0$

Hægt er að sýna að $f(\pm 1) < 0$ → tvær rætur finnast innan bils $[-1,+1]$



→ ein rót verður utan þess bils, sem leiðir til þvergildra lausna fyrir θ því $\cos\theta \geq 1$

① Dæmi 12-03 í bók

(1)

Kerfi tveggja tengdra hreintóna sveifla, lýst með

$$\begin{cases} \ddot{x}_1 + \frac{m}{M}\ddot{x}_2 + \omega_0^2 x_1 = 0 \\ \ddot{x}_2 + \frac{m}{M}\ddot{x}_1 + \omega_0^2 x_2 = 0 \end{cases}$$

Setjum

$$\bar{x}(t) = \begin{pmatrix} B_1 e^{i\omega t} \\ B_2 e^{i\omega t} \end{pmatrix} \;\rightarrow\; \ddot{\bar{x}}(t) = -\omega^2 \begin{pmatrix} B_1 e^{i\omega t} \\ B_2 e^{i\omega t} \end{pmatrix} = -\omega^2 \bar{x}(t)$$

→ Umskrifum þetta sem hneppi

$$\begin{pmatrix} 1 & \frac{m}{M} \\ \frac{m}{M} & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \omega_0^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

(2)

því fæst

$$\begin{pmatrix} 1 & \frac{m}{M} \\ \frac{m}{M} & 1 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{\omega_0^2}{\omega^2}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

Sem er greinilega eigingildis verkefni

sem hefur eigingildi

$$\frac{\omega_0^2}{\omega^2} = 1 - \frac{m}{M} \quad \text{með eiginvigur} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega = \frac{\omega_0}{\sqrt{1 - \frac{m}{M}}}$$

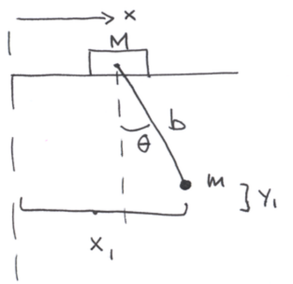
$$\frac{\omega_0^2}{\omega^2} = 1 + \frac{m}{M} \quad \text{með eiginvigur} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega = \frac{\omega_0}{\sqrt{1 + \frac{m}{M}}}$$

Andsamhverfur

samhverfur sveifluháttur
Lægri tíðni

② Dæmi 12-18 í bók

(3)



Notum kortísk hnit ~~til að~~ raða saman

$$x_1 = x + b\sin\theta \qquad y_1 = b - b\cos\theta$$

$$\downarrow \qquad\qquad \downarrow$$

$$\dot{x}_1 = \dot{x} + \dot{\theta} b\cos\theta \qquad \dot{y}_1 = \dot{\theta} b\sin\theta$$

$$\rightarrow T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left\{\dot{x}_1^2 + \dot{y}_1^2\right\}$$

$$= \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left\{(\dot{x} + \dot{\theta} b\cos\theta)^2 + (\dot{\theta} b\sin\theta)^2\right\}$$

$$= \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left\{\dot{x}^2 + (b\dot{\theta})^2 + 2b\dot{x}\dot{\theta}\cos\theta\right\}$$

(4)

$$U = mgy_1 = mgb\{1 - \cos\theta\} \simeq mgb\left\{1 - 1 + \frac{\theta^2}{2} + \cdots\right\} = \frac{mgb}{2}\theta^2$$

Tökum saman liði í T

$$T = \left(\frac{M+m}{2}\right)\dot{x}^2 + \frac{m}{2}\left((b\dot{\theta})^2 + 2b\dot{x}\dot{\theta}\right)$$

fyrir lítið horn $\cos\theta \sim 1 + \cdots$

$$M = \begin{pmatrix} \frac{\partial^2 T}{\partial \dot{x}^2} & \frac{\partial^2 T}{\partial \dot{x}\partial\dot{\theta}} \\ \frac{\partial^2 T}{\partial\dot{\theta}\partial\dot{x}} & \frac{\partial^2 T}{\partial\dot{\theta}^2} \end{pmatrix} = \begin{pmatrix} M+m & mb \\ mb & mb^2 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\partial\theta} \\ \frac{\partial^2 U}{\partial x\partial\theta} & \frac{\partial^2 U}{\partial\theta^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & mgb \end{pmatrix}$$

ekki sama vídd

(5)

$A\bar{a} = \omega^2 M \bar{a}$ — hér tapast samhverfa ef reynt er að margfalda með M^{-1}

Lítum því á jöfnuknippið

$$(A - \omega^2 M)\bar{a} = 0 \quad \longleftrightarrow \quad \begin{pmatrix} -\omega^2(M+m) & -\omega^2 mb \\ -\omega^2 mb & mgb - \omega^2 mb^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

Svo lausn sé til verður ákveðan að hverfa

$$\omega^2 (M+m)\{\omega^2 mb^2 - mgb\} - \omega^4 m^2 b^2 = 0$$

$$\omega^2\{\omega^2 M b^2 m - mgb(m+M)\} = 0 \quad \Big\} \longrightarrow \quad \omega_1 = 0, \quad \omega_2 = \sqrt{\frac{g}{Mb}(M+m)}$$

(6)

Eiginvigur ω_1 verður að vera $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Fyrir ω_2:

$$\begin{pmatrix} -\omega_2^2 (M+m) & -\omega_2^2 mb \\ -\omega_2^2 mb & mgb - \omega_2^2 mb^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$= \omega_2^2 \begin{pmatrix} -(M+m) & -mb \\ -mb & -\frac{m^2 b^2}{M+m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \leadsto \quad \begin{pmatrix} -\frac{bm}{M+m} \\ 1 \end{pmatrix} \text{ (eiginvigur)}$$

$$\longrightarrow \quad a_1 = -\frac{bm}{M+m} a_2$$

(7)

$$U = \begin{pmatrix} 1 & -\frac{bm}{M+m} \\ 0 & 1 \end{pmatrix}$$

$$\bar{a} = \begin{pmatrix} x \\ \theta \end{pmatrix} = U\bar{\eta} = \begin{pmatrix} \eta_1 - \eta_2 \frac{bm}{M+m} \\ \eta_2 \end{pmatrix}$$

$$\left.\begin{array}{l} \longrightarrow \eta_2 \sim \theta \\ \eta_1 - \eta_2 \frac{bm}{M+m} \sim x \end{array}\right\} \longrightarrow \quad \eta_1 \sim x + \frac{bm}{M+m}\theta$$

Sveifluháttur η_2 verður þegar $\eta_1 = 0$ eða $x = -\frac{bm}{M+m}\theta$

η_1 verður þegar $\eta_2 = 0$ eða $\theta = 0$

(8)

③ Dæmi 12-23 í bók

Sýna að orka eiginsveifluhátts sé fasti
sýna fyrir síðan tyi 12·03

Eiginsveifluháttur $\eta_r = \beta_r e^{i\omega_r t} \longrightarrow \dot{\eta}_r = i\omega_r \beta_r e^{i\omega_r t} = i\omega_r \eta_r$

Orkan

$$E_r = T_r + U_r = \frac{1}{2}\dot{\eta}_r^2 + \frac{1}{2}\omega_r^2 \eta_r^2$$

Annars stig, eða annað veldið af η_r þarf alltaf að taka með varúð, Hér er heppilegt að nota raunhlutann $\eta_r^2 \longrightarrow \{\mathrm{Re}\,\eta_r\}^2$

þ.a.

$$E_r = \frac{1}{2}(\mathrm{Re}\,\dot{\eta}_r)^2 + \frac{1}{2}\omega_r^2(\mathrm{Re}\,\eta_r)^2$$

$$(\mathrm{Re}\,\eta_r)^2 = \left\{\mathrm{Re}\left[(\beta_r' + i\beta_r'')(\cos(\omega_r t) + i\sin(\omega_r t))\right]\right\}^2$$

$$= \left\{\beta_r'\cos(\omega_r t) - \beta_r''\sin(\omega_r t)\right\}^2$$

$$(\mathrm{Re}\,\dot{\eta}_r)^2 = \left\{\mathrm{Re}\left[i\omega_r(\beta_r' + i\beta_r'')(\cos(\omega_r t) + i\sin(\omega_r t))\right]\right\}^2$$

$$= \left\{-\omega_r\left(\beta_r''\cos(\omega_r t) + \beta_r'\sin(\omega_r t)\right)\right\}^2$$

$$\to E_r + T_r + U_r = \frac{1}{2}\omega_r^2\left\{(\beta_r')^2 + (\beta_r'')^2\right\}$$

$$= \frac{1}{2}\omega_r^2|\beta_r|^2$$

Ég lét það nægja, því sveifluhættirnir eru hornréttir → orka flyst ekki milli þeirra. Samanber t.d. eiginástand jöfnu Schrödingers í skammtafræði. . .

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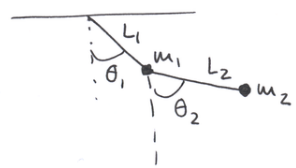
④ Dæmi 12-27 í bók

Tvöfaldur sveifill



Við höfum leitt út áður

$$T = \frac{m_1}{2}(L_1\dot{\theta}_1)^2 + \frac{m_2}{2}\left\{(L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right\}$$

$$\approx \frac{\dot{\theta}_1^2}{2}\left\{m_1L_1^2 + m_2L_2^2\right\} + \frac{m_2}{2}(L_2\dot{\theta}_2)^2 - m_2L_1L_2\dot{\theta}_1\dot{\theta}_2$$

$$\to M = \begin{pmatrix} (m_1+m_2)L_1^2 & -m_2L_1L_2 \\ -m_2L_1L_2 & m_2L_2^2 \end{pmatrix}$$

$$U = m_1gL_1\{1 - \cos\theta_1\} + m_2g\{L_1(1-\cos\theta_1) + L_2(1-\cos\theta_2)\}$$

$$\approx \{m_1 + m_2\}gL_1\frac{\theta_1^2}{2} + m_2gL_2\frac{\theta_2^2}{2}$$

og því

$$A = \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{pmatrix}$$

$$(A - \omega^2 M)\bar{a} = 0$$

$$\begin{pmatrix} (m_1+m_2)gL_1 - \omega^2(m_1+m_2)L_1^2 & +\omega^2 m_2L_1L_2 \\ +\omega^2 m_2L_1L_2 & m_2gL_2 - \omega^2 m_2L_2^2 \end{pmatrix}\bar{a} = 0$$

(13)

Skrifum sem

$$\begin{pmatrix} A-\omega^2 B & \omega^2 x \\ \omega^2 x & C-\omega^2 D \end{pmatrix} \rightarrow \det\left(A-\omega^2 M\right)$$

$$= (A-\omega^2B)(C-\omega^2D) - \omega^4 x^2 = 0$$

$$\rightarrow \omega^4(x^2-BD) + \omega^2(BC+AD) - AC = 0$$

$$\rightarrow \omega^2 = \frac{-BC+AD \pm \sqrt{(BC+AD)^2 + 4(x^2-BD)AC}}{2(x^2-BD)}$$

$$x^2 - BD = (m_2L_1L_2)^2 - (m_1+m_2)L_1^2 m_2 L_2^2 = -m_1 m_2 L_1^2 L_2^2$$

$$BC+AD = (m_1+m_2)L_1^2 m_2 g L_2 + (m_1+m_2) g L_1 m_2 L_2^2$$

$$= (m_1+m_2) m_2 g \left(L_1^2 L_2 + L_1 L_2^2\right)$$

athugið víddina

$$\rightarrow \omega_\pm^2 = \frac{(m_1+m_2)g(L_1+L_2) \pm \sqrt{(m_1+m_2)g^2\left\{m_1(L_1-L_2)^2 + m_2(L_1+L_2)^2\right\}}}{2m_1L_1L_2}$$

(14)

$$\begin{pmatrix} A-\omega_\pm^2 B & \omega_\pm^2 x \\ \omega_\pm^2 x & C-\omega_\pm^2 D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\left.\begin{aligned} (A-\omega_\pm^2 B)a + \omega_\pm^2 x b &= 0 \\ \omega_\pm^2 x a + (C-\omega_\pm^2 D) b &= 0 \end{aligned}\right\} \rightarrow \left.\frac{a}{b}\right|_\pm = -\frac{\omega_\pm x}{(A-\omega_\pm^2 B)}$$

Hér er hægt að sýna að $(A-\omega_+^2B) < 0$ og $(A-\omega_-^2B) > 0$

því er hægt að sýna að

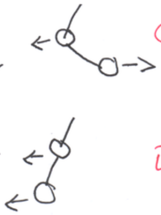
jákvæð stærð

$$\nu_+ \sim \theta_1 - \cdots\theta_2$$

$$\nu_- \sim \theta_1 + \cdots\theta_2$$

$\rightarrow$ ν_+ er — andfasi

ν_- er — í fasa



(15)

(5) Dæmi 12-08 í bók

Notum sama dæmi aftur, en einfaldara því $m_1 = m_2$, $L_1 = L_2$

köllum m og l

$$\rightarrow A = mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad M = ml^2 \begin{pmatrix} 2 & +1 \\ +1 & 1 \end{pmatrix}$$

$$\det(A-\omega^2M) = \begin{vmatrix} 2mgl - 2ml^2\omega^2 & -\omega^2 ml^2 \\ -\omega^2 ml^2 & mgl - ml^2\omega^2 \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} \frac{2g}{l} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{l} - \omega^2 \end{vmatrix} = 0$$

$$\rightarrow \left(\frac{2g}{l} - 2\omega^2\right)\left(\frac{g}{l} - \omega^2\right) - \omega^4 = 0$$

7-07 í bók

$$T = \frac{ml^2}{2}\left(2\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\right)$$

$$U = \frac{mgl}{2}\left(2\phi_1^2 + \phi_2^2\right)$$

(16)

$$\rightarrow 2\left(\frac{g}{l}\right)^2 - \omega^2\left(\frac{2g}{l} + \frac{2g}{l}\right) + 2\omega^4 - \omega^4 = 0$$

$$\rightarrow \omega^4 - 4\frac{g}{l}\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

$$\rightarrow \omega^2 = \frac{4\frac{g}{l} \pm \sqrt{\left(16\left(\frac{g}{l}\right)^2 - 8\left(\frac{g}{l}\right)^2\right)}}{2} = \left\{2 \pm \sqrt{2}\right\}\frac{g}{l}$$

$$\rightarrow \omega_1 = \sqrt{2+\sqrt{2}}\sqrt{\frac{g}{l}}$$

$$\omega_2 = \sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}$$

þurfum að leysa

$$\left(A - \omega_{1,2}^2 M\right)\bar{a} = 0$$

Hér fæst fyrir ω_1

$$a_2 = -\sqrt{2}\, a_1$$

og fyrir ω_2

$$a_2 = \sqrt{2}\, a_1$$

eiginvigrar $\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} : \omega_1$ $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} : \omega_2$

$$\rightarrow U = \begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix} \qquad U^t = \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix} \qquad \eta = U^t \bar{a} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

↑•↓
↓•↑

$\rightarrow \eta_1 \sim x_1 - \sqrt{2}\, x_2$ η_1 gerist þ. $\eta_2 = 0$ þ.e. $x_1 = \sqrt{2}\, x_2$

$\eta_2 \sim x_1 + \sqrt{2}\, x_2$ η_2 gerist þ. $\eta_1 = 0$ þ.e. $x_1 \sim -\sqrt{2}\, x_2$

↑•↓
↑•↓

(17)

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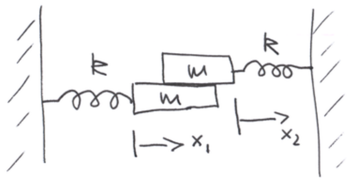
(6) Dæmi 12-06 í bók

(18)



núningskraftur á 1 vegna 2

$$f = -\beta(\dot{x}_1 - \dot{x}_2)$$

$$\rightarrow \begin{cases} m\ddot{x}_1 + \beta(\dot{x}_1 - \dot{x}_2) + Kx_1 = 0 \\ m\ddot{x}_2 + \beta(\dot{x}_2 - \dot{x}_1) + Kx_2 = 0 \end{cases}$$

ekki geymið kerfi

Reynum lausnir $x_1(t) = A e^{\alpha t}$ $x_2(t) = B e^{\alpha t}$, $\alpha \in \mathbb{C}$

$$m\alpha^2 A + \beta\alpha(A - B) + KA = 0$$

$$m\alpha^2 B + \beta\alpha(B - A) + KB = 0$$

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(19)

Þá

$$\begin{pmatrix} m\alpha^2 + K + \beta\alpha & -\beta\alpha \\ -\beta\alpha & m\alpha^2 + K + \beta\alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

ákveðan

$$\rightarrow (m\alpha^2 + \beta\alpha + K)^2 - (\beta\alpha)^2 = 0$$

$$\rightarrow m\alpha^2 + \beta\alpha + K = \pm\beta\alpha \quad \rightarrow m\alpha^2 + K = 0$$

eða $m\alpha^2 + 2\beta\alpha + K = 0$

$$\alpha^2 = -\frac{K}{m} \longrightarrow \alpha_1 = \pm i\sqrt{\frac{K}{m}}$$

$$\alpha^2 + 2\frac{\beta}{m}\alpha + \frac{K}{m} = 0 \rightsquigarrow \alpha_2 = \frac{1}{m}\left\{-\beta \pm \sqrt{\beta^2 - Km}\right\}$$

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(20)

α_1 - hátturinn er sveiflulaus með $\omega_1 = \sqrt{\frac{K}{m}}$

lausnin er því

$$x_1(t) = A_1^+ e^{i\omega_1 t} + A_1^- e^{-i\omega_1 t} + e^{-\frac{\beta}{m}t}\left\{B_1^+ e^{+\frac{\sqrt{\beta^2 - Km}}{m}t} + B_1^- e^{-\frac{\sqrt{\beta^2 - Km}}{m}t}\right\}$$

$$x_2(t) = A_2^+ e^{i\omega_1 t} + A_2^- e^{-i\omega_1 t} + e^{-\frac{\beta}{m}t}\left\{B_2^+ e^{+\frac{\sqrt{\beta^2 - Km}}{m}t} + B_2^- e^{-\frac{\sqrt{\beta^2 - Km}}{m}t}\right\}$$

sveiflandi Eiginþættir

→ engin innbyrðis hreyfing

→ í fasa

deyfður háttur

→ í andfasa