③ Dæmi 8-02 í bók.

Heildið

$$\theta(r) = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu\left\{E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right\}}} + \overset{\text{fasti}}{C}$$

Í bókinni er bent á breytuskipti $u = \frac{1}{r}$, $du = -\frac{1}{r^2}dr$

$$\to \theta = -\int \frac{du\cdot \ell}{\sqrt{2\mu\left\{E + uk - \frac{u^2\ell^2}{2\mu}\right\}}} = -\int \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu k}{\ell^2}u - u^2}}$$

Notum (E.8b) eða (GR-2.261)

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{-a}}\arcsin\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right) \qquad \begin{cases} a<0 \\ b^2 > 4ac \\ |2ax+b| < \sqrt{b^2-4ac} \end{cases}$$

$$\to \theta + \theta_0 = +\arcsin\left[\frac{-2u + \frac{2\mu k}{\ell^2}}{\sqrt{\left(\frac{2\mu k}{\ell^2}\right)^2 + \frac{8\mu E}{\ell^2}}}\right]$$

$$\to \sin(\theta + \overset{\text{heildunarfasti}}{\theta_0}) = \frac{-\frac{2}{r} + \frac{2\mu k}{\ell^2}}{\sqrt{\left(\frac{2\mu k}{\ell^2}\right)^2 + \frac{8\mu E}{\ell^2}}}$$

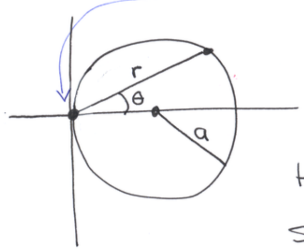
Notum $\sin(\theta - \frac{\pi}{2}) = -\cos\theta$, þ.e. veljum $\theta_0 = -\frac{\pi}{2}$

$$\to \cos\theta = \frac{-\frac{2\mu k}{\ell^2} + \frac{2}{r}}{\sqrt{\left(\frac{2\mu k}{\ell^2}\right)^2 + \frac{8\mu E}{\ell^2}}} = \frac{\frac{\ell^2}{\mu k r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}$$

② Dæmi 8.11 í bók

"Ögn í kraftsviði $F(r) = -\frac{k}{r^n}$

Ef brautin er hringbraut í gegnum kraftmiðju, sýnið að $n = 5$



Hreyfijafnan (8.21) er

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

Hægt er að stika hringinn miðað við kraftmiðju sem $r = 2a\cos\theta$, sú braut ætti að geta kraftinn, reynum

$$\frac{1}{r} = \frac{1}{2a\cos\theta} \to \frac{d^2}{d\theta^2}\left(\frac{1}{2a\cos\theta}\right) = \frac{\sin^2\theta}{a\cos^3\theta} + \frac{1}{2a\cos\theta}$$

(reiknað með maxima)

Setjum inn í hreyfijöfnu

$$\to \frac{\sin^2\theta}{a\cos^3\theta} + \frac{1}{2a\cos\theta} + \frac{1}{2a\cos\theta} = -\frac{\mu}{\ell^2}\left\{(2a)^2\cos^2\theta\right\}F$$

$$\to \frac{\sin^2\theta}{\cos^5\theta} + \frac{1}{\cos^3\theta} = -\frac{\mu}{\ell^2}4a^3 F$$

$$\to \frac{\sin^2\theta + \cos^2\theta}{\cos^5\theta} = -\frac{\mu}{\ell^2}4a^3 F$$

$$\to \frac{1}{\cos^5\theta} = -\frac{\mu}{\ell^2}4a^3 F \to F = -\frac{\ell^2}{4a^3\mu}\frac{1}{\cos^5\theta}$$

$$\to \boxed{F = -\frac{\ell^2}{4a^3\mu}\frac{(2a)^5}{\{2a\cos\theta\}^5} = -\frac{8a^2\ell^2}{\mu}\frac{1}{r^5}}$$

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① Dæmi 8-08 í bók

"Ögn er hrint frá kraftmiðju með $F(r) = kr$

$\bar{F} = -\nabla U \;\to\; U = -\frac{kr^2}{2}$, sýna að brautin sé gleiðbogi.

Lausn hreyfijöfnunnar er (8.17)

$$\theta(r) = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}} = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu\left(E + \frac{kr^2}{2} - \frac{\ell^2}{2\mu r^2}\right)}}$$

Reiknum breytuskipti $x = r^2 \to dx = 2r\,dr$, $\frac{dr}{r^2} = \frac{dx}{r^2 2r} = \frac{1}{2}\frac{dx}{x}\frac{1}{\sqrt{x}}$

$$\to \theta = \frac{1}{2}\int \frac{dx}{x\sqrt{\frac{2\mu E}{\ell^2}x + \frac{\mu k}{\ell^2}x^2 - 1}} = \frac{1}{2}\int \frac{dx}{x\sqrt{\frac{\mu k}{\ell^2}x^2 + \frac{2\mu E}{\ell^2}x - 1}}$$

6

Heildið er byggjanlegt skv. (E.10b) sjá (GR-2.266)

$$\theta = \frac{1}{2}\arcsin\left\{\frac{\frac{\mu E}{\ell^2}x - 1}{x\sqrt{\left(\frac{\mu E}{\ell^2}\right)^2 + \frac{\mu k}{\ell^2}}}\right\} + \theta_0$$

$$\to \sin\{2(\theta-\theta_0)\} = \frac{\frac{\mu E}{\ell^2}}{\sqrt{\left(\frac{\mu E}{\ell^2}\right)^2 + \frac{\mu k}{\ell^2}}} - \frac{1}{r^2\sqrt{\left(\frac{\mu E}{\ell^2}\right)^2 + \frac{\mu k}{\ell^2}}}$$

$$= \frac{1}{\sqrt{1 + \frac{k\ell^2}{\mu E^2}}} - \frac{1}{r^2}\frac{\frac{\ell^2}{\mu E}}{\sqrt{1 + \frac{k\ell^2}{\mu E^2}}} \equiv \alpha - \frac{\beta}{r^2}$$

$\sin\{2(\theta-\theta_0)\} = \alpha - \frac{\beta}{r^2}$, Ef við höfðum $\cos(2\theta)$ þá mætti nota $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

setjum því $\theta_0 = \frac{\pi}{4}$

7

$$\sin\left\{2\theta - \frac{\pi}{2}\right\} = -\cos(2\theta) = \alpha - \frac{\beta}{r^2} \;\to\; r^2\cos(2\theta) + r^2\alpha = \beta$$

$$\to r^2\{\cos^2\theta - \sin^2\theta\} + r^2\alpha = \beta$$

$$x^2 - y^2 + (x^2+y^2)\alpha = \beta$$

$$\boxed{x^2(1+\alpha) - y^2(1-\alpha) = \beta}$$

$$\alpha = \frac{1}{\sqrt{1 + \frac{k\ell^2}{\mu E^2}}} < 1$$

$$\to 1-\alpha > 0$$

→ svo þetta er jafna gleiðboga

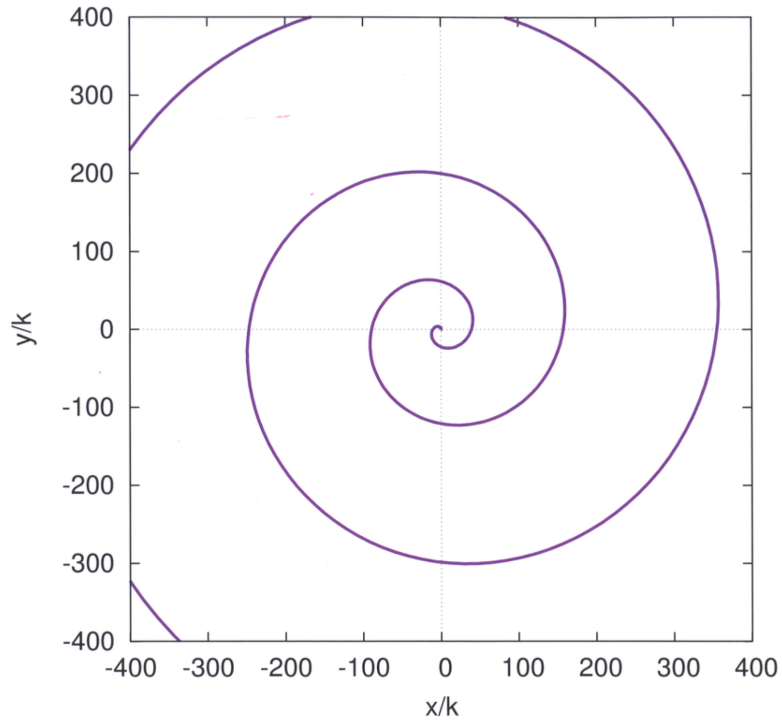
④ Dæmi 8-14 í bók

Braut gefin sem $r = k\theta^2$ — fasti

Finna kraftinn

sjá mynd á næstu síðu

7b



Notum hreyfijöfnu (8.21)

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)}$$

$$\frac{1}{r} = \frac{1}{k\theta^2} \to \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{6k}{r^2}$$

$$\to \frac{6k}{r^2} + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$-\left\{\frac{6k}{r^4} + \frac{1}{r^3}\right\}\frac{\ell^2}{\mu} = F(r)$$

(5) Dæmi 8-22 í bók

Skoðum hreyfingu í krafti

$$F(r) = -\frac{k}{r^3} \to U(r) = -\frac{k}{2r^2}$$

Virka mættið

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$= \frac{1}{2}\left\{\frac{\ell^2}{\mu} - k\right\}\frac{1}{r^2}$$

miðflótta kraftur og mættið hafa sama form

Schrödinger - - - - - -

Athugum hreyfijöfnuna

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2u^2}F(1/u) = -\frac{\mu}{\ell^2u^2}\{-ku^3\} = +\frac{\mu k}{\ell^2}u$$

$$\to \boxed{\frac{d^2u}{d\theta^2} + \left\{1 - \frac{\mu k}{\ell^2}\right\}u = 0}$$

Skoðum tilfellin sem koma til greina

(1) $\ell^2 = \mu k$ (ℓ tengist hverfiþunga)

$\to \frac{d^2u}{d\theta^2} = 0$ með lausn $u = A\theta + B$

$u = \frac{1}{r} \to r = \frac{1}{A\theta + B}$ gormhreyfing að miðju

(2) $\ell^2 > \mu k \to \left\{1 - \frac{\mu k}{\ell^2}\right\} = \gamma^2 > 0$

$$\to \boxed{\frac{d^2u}{d\theta^2} + \gamma^2 u = 0}$$

með lausn $u = A\cos(\gamma\theta + \delta) \to r = \frac{1}{A\cos(\gamma\theta+\delta)}$

$u \in [-1, 1] \to Ar \in \mathbb{R}$ og brautin er opin

(3) $\ell^2 < \mu k \to \left\{1 - \frac{\mu k}{\ell^2}\right\} = -\kappa^2$, með $\kappa^2 > 0$

$$\to \boxed{\frac{d^2u}{d\theta^2} - \kappa^2 u = 0} \to u = A\cosh(\kappa\theta + \delta)$$

$$r = \frac{1}{A\cosh(\kappa\theta + \delta)} \qquad \cosh x \in [1, \infty)$$

$\to Ar \in (0, 1]$ hreyfing að miðju

Stöðugleiki hringbrautar

$g(r) = \frac{1}{\mu}\frac{\partial U}{\partial r} = \frac{k}{\mu r^3}$, $r \to \rho + x$ þ.s. $x \ll \rho$

frávik frá hring

$$\ddot{x} - \frac{\ell^2}{\mu^2\rho^3\left[1 + \frac{x}{\rho}\right]^3} = -g(\rho + x)$$

$$= -\frac{k}{\mu\rho^3\left[1 + \frac{x}{\rho}\right]^3}$$

$$\to \boxed{\ddot{x} + \left\{k - \frac{\ell^2}{\mu}\right\}\frac{1}{\mu\rho^3\left(1 + \frac{x}{\rho}\right)^3} = 0}$$

$$g(\rho) = \frac{\ell^2}{\mu^2\rho^3} \to k = \frac{\ell^2}{\mu}$$

$\ddot{r}\big|_{r=\rho} = 0 \to k = \frac{\ell^2}{\mu}$ fyrir stöðuga hringbraut

$\to \ddot{x} = 0 \Leftarrow$ en þessi jafna gefur ekki takmarkaða lausn $\to$ engin stöðug hringbraut

6 Dømi 8-21 í bók

$E = \text{fasti}$

Ef støðug hringbraut, þá er l^2 størra en fyrir fyrir aðra braut

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$

$$\longrightarrow \quad l^2 = 2\mu r^2 \left\{ E - U(r) - \frac{\mu}{2}\dot{r}^2 \right]$$

Sama E og U, hringbraut $\longrightarrow \dot{r} = 0$

fyrir allar aðrar brautir $\dot{r} \neq 0$

$\longrightarrow$ $\boxed{\text{þessi liður dregur úr } l^2}$