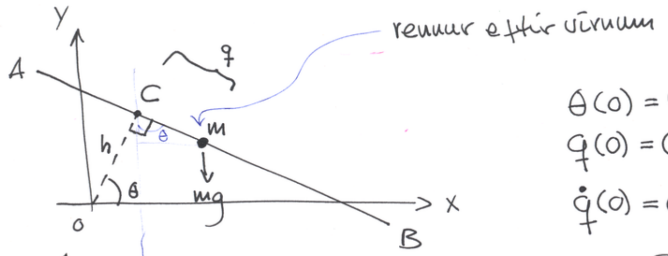
① Dæmi 7-17 í bók

①



$\theta(0) = 0$

$q(0) = 0$

$\dot{q}(0) = 0$

Sjáum ~~að~~ $q(t) = \frac{g}{2\omega^2}\left\{\cosh(\omega t) - \cos(\omega t)\right\}$

snýst í sléttunni $\dot{\theta} = \omega = $ fasti um O

$$x = h\cos\theta + q\sin\theta \qquad \theta = \omega t$$

$$y = h\sin\theta - q\cos\theta \qquad q = q(t)$$

$$\rightarrow \quad \dot{x} = -h\omega\sin(\omega t) + q\omega\cos(\omega t) + \dot{q}\sin(\omega t)$$

$$\dot{y} = h\omega\cos(\omega t) + q\omega\sin(\omega t) - \dot{q}\cos(\omega t)$$

②

$$\rightarrow T = \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2\right\} = \frac{m}{2}\left\{(h\omega)^2 + (q\omega)^2 + \dot{q}^2 - 2h\omega\dot{q}\right\}$$

allir aðrir „blandaðir liðir" hverfa vegna formerkja

$$U = mgy = mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\}$$

því fæst

$$L = \frac{m}{2}\left\{(h\omega)^2 + (q\omega)^2 + \dot{q}^2 - 2h\omega\dot{q}\right\} - mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\}$$

Aðeins eitt alhnit, q

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \rightarrow m\omega^2 q + mg\cos(\omega t) - \frac{d}{dt}\left\{m\dot{q} - mh\omega\right\} = 0$$

$$\rightarrow \boxed{\ddot{q} - \omega^2 q = g\cos(\omega t)}$$

③

„Samskonar" jafna og í 1. dæmi á síðasta dæmablaði

$$q(t) = Ae^{\omega t} + Be^{-\omega t} - \frac{g}{2\omega^2}\cos(\omega t)$$

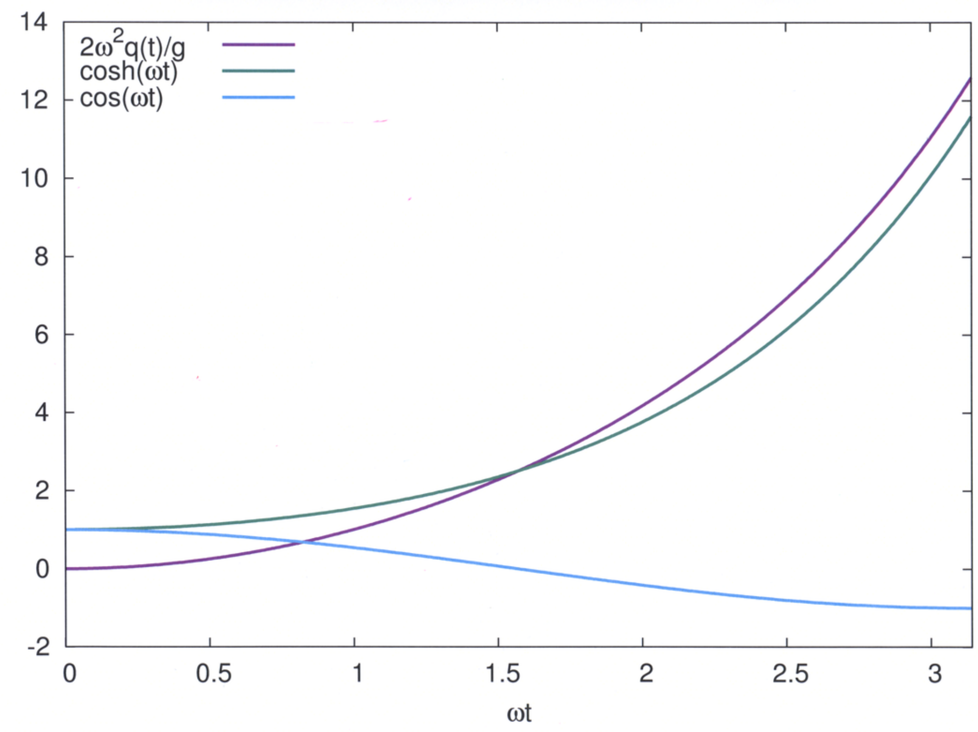
Athugum upphafsskilyrðin

$$\left.\begin{aligned} q(0) &= A + B - \frac{g}{2\omega^2} = 0 \\ \dot{q}(0) &= A - B = 0 \end{aligned}\right\} \rightarrow \quad \begin{aligned} A &= \frac{g}{4\omega^2} \\ B &= -\frac{g}{4\omega^2} \end{aligned}$$

$$\rightarrow q(t) = \frac{g}{2\omega^2}\left\{\cosh(\omega t) - \cos(\omega t)\right\}$$

sjá mynd á næstu síðu af q(t).

④



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Finnum fall Hamiltons

$$p = \frac{\partial L}{\partial \dot q} = m\dot q - mh\omega \;\longrightarrow\; \dot q = \frac{p}{m} + h\omega$$

$$H = p\dot q - L = m\dot q^2 - mh\omega\dot q - L$$

$$= \frac{m}{2}\left\{\dot q^2 - (h\omega)^2 - (q\omega)^2\right\} + mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\}$$

$$= \frac{p^2}{2m} + ph\omega - \frac{1}{2}m(q\omega)^2 + mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\}$$

$$E = T + U = \frac{p^2}{2m} + \frac{1}{2}m(q\omega)^2 + mg\left\{h\sin(\omega t) - q\cos(\omega t)\right\} \neq H$$

$H = H(t)$ og því ekki heildarorkan, snúningur í þyngdarsviði leiðir til orkutilfærslu

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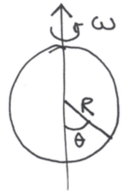
3 Dæmi 7-21 í bók



Eitt alhnit, θ

Snúningshreyfing með ferð ω

$$T = \underbrace{\frac{m}{2}(R\dot\theta)^2}_{\text{hreyfing á hring}} + \frac{m}{2}(R\sin\theta)^2\omega^2, \quad U = -mgR\cos\theta$$

Finna jafnvægisstöður sínar og smáar sveiflur um hana

Fyrst, hreyfijafna

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0 \;\rightarrow\; m(R\omega)^2\sin\theta\cdot\cos\theta - mgR\sin\theta - \frac{d}{dt}\left\{mR^2\dot\theta\right\} = 0$$

$$\rightarrow \boxed{\ddot\theta - \omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0}$$

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Jafnvægisstöður eru þegar „kraftar jafnast út" eða þess vegna þegar hröðun er hverfandi

þegar $\ddot\theta\big|_{\theta_i} = 0$ Hreyfijafnan gefur

$$\ddot\theta\big|_{\theta_i} = \left\{\omega^2\cos\theta_i - \frac{g}{R}\right\}\sin\theta_i = 0$$

Sem hefur 3 lausnir

$$\theta_1 = 0, \quad \theta_2 = \pi, \quad \theta_3 = \arccos\left\{\frac{g}{\omega^2 R}\right\}$$

Skilgreinum $\delta_i = \theta - \theta_i$ og fáum hreyfijöfnu

$$\ddot\delta_i - \omega^2\left\{\cos(\theta_i + \delta_i) - \frac{g}{\omega^2 R}\right\}\sin(\theta_i + \delta_i) = 0$$

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$$\sin(\theta_i + \delta_i) = \sin\theta_i\cdot\cos\delta_i + \cos\theta_i\cdot\sin\delta_i \approx \sin\theta_i + \delta_i\cos\theta_i$$

$$\cos(\theta_i + \delta_i) = \cos\theta_i\cdot\cos\delta_i - \sin\theta_i\sin\delta_i \simeq \cos\theta_i - \delta_i\sin\theta_i$$

því er hreyfijafnan

$$\ddot\delta_i - \omega^2\left\{\cos\theta_i - \frac{g}{\omega^2 R} - \delta\sin\theta_i\right\}\left\{\sin\theta_i + \delta_i\cos\theta_i\right\} \approx 0$$

$\theta_1 = 0$

$$\rightarrow \boxed{\ddot\delta_1 - \omega^2\left\{1 - \frac{g}{\omega^2 R}\right\}\delta = 0}$$

stöðugt ef $\omega^2 < \frac{g}{R}$ með tíðni $\omega_1 = \sqrt{\omega^2 - \frac{g}{R}}$

(9)

$\theta_2 = \pi$

$$\rightarrow \boxed{\ddot{\delta}_2 - \omega^2\left\{1 + \frac{g}{\omega^2 R}\right\}\delta_i = 0}$$

sem er alltaf óstöðug „sveifla" því $\omega_2^2 = -\left\{\omega^2 + \frac{g}{R}\right\} < 0$

$\theta_3 = \arccos\left\{\frac{g}{\omega^2 R}\right\}$

$$\rightarrow \ddot{\delta}_3 + \omega^2 \sin\theta_3 \cdot \delta_3 \cdot \sin\theta_3 = 0$$

$$\boxed{\ddot{\delta}_3 + \omega^2 \sin^2\theta_3 \cdot \delta_3 = 0}$$

$$\sin^2\theta_3 = 1 - \cos^2\theta_3 = 1 - \frac{g^2}{R^2\omega^4}$$

$$\rightarrow \omega_3 = \sqrt{\omega^2 - \left(\frac{g}{R\omega}\right)^2} = \omega^2\sqrt{1 - \left(\frac{g}{R\omega^2}\right)^2}$$

(10)

skilgreinum marktíðni $\omega_c^2 = \frac{g}{R}$

$$\rightarrow \omega_1 = \sqrt{\omega^2 - \omega_c^2} = \omega\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$ stöðug $\omega < \omega_c$

$$\omega_3 = \sqrt{\omega^2 - \frac{\omega_c^4}{\omega^2}} = \omega\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^4}$$ stöðug $\omega \geq \omega_c$

Finnum H

$$H = p\dot{q} - L = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$

$$= mR^2\dot{\theta}^2 - \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}(\omega R\sin\theta)^2 + mgR\cos\theta$$

$$= \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}(\omega R\sin\theta)^2 + mgR\cos\theta$$

$$\rightarrow \frac{H}{mR^2} = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}(\omega\sin\theta)^2 + \frac{g}{R}\cos\theta$$

$$\rightarrow \frac{H}{mR^2\omega_c^2} = \frac{1}{2}\left\{\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\sin\theta\right)^2\right\} - \cos\theta$$

(11)

því er fasarit fyrir fast $\frac{H}{mR^2\omega_c^2} = \kappa$

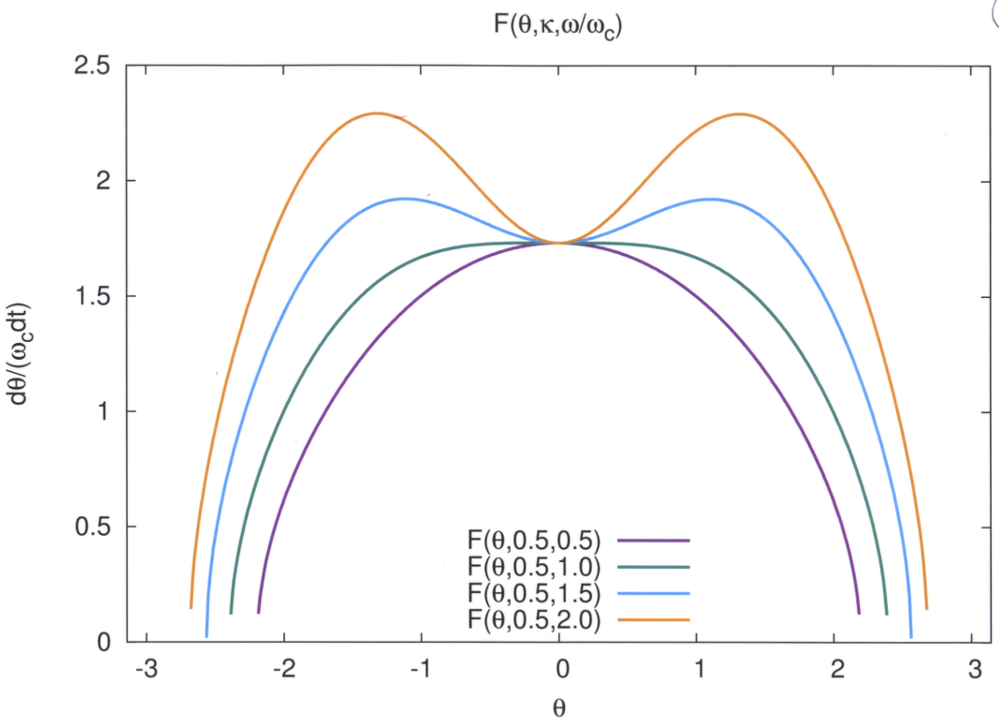
$$\left(\frac{\dot{\theta}}{\omega_c}\right)^2 = 2\left\{\kappa + \cos\theta\right\} + \left(\frac{\omega}{\omega_c}\right)^2\sin^2\theta$$

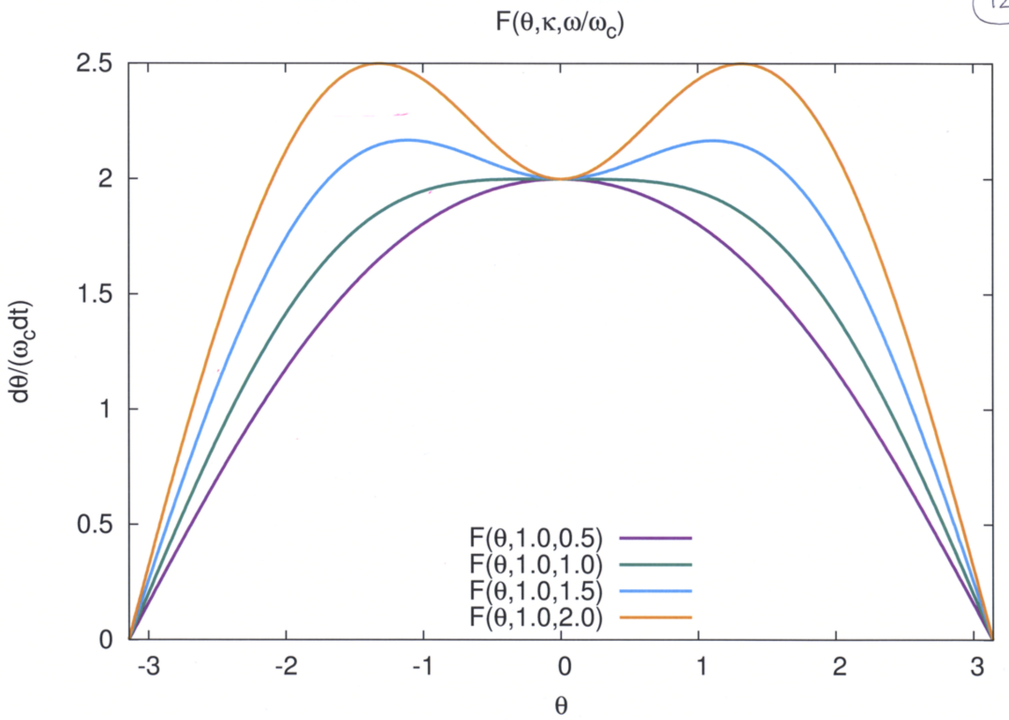
$$\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\sin\theta\right)^2 = 2\left\{\kappa + \cos\theta\right\}$$

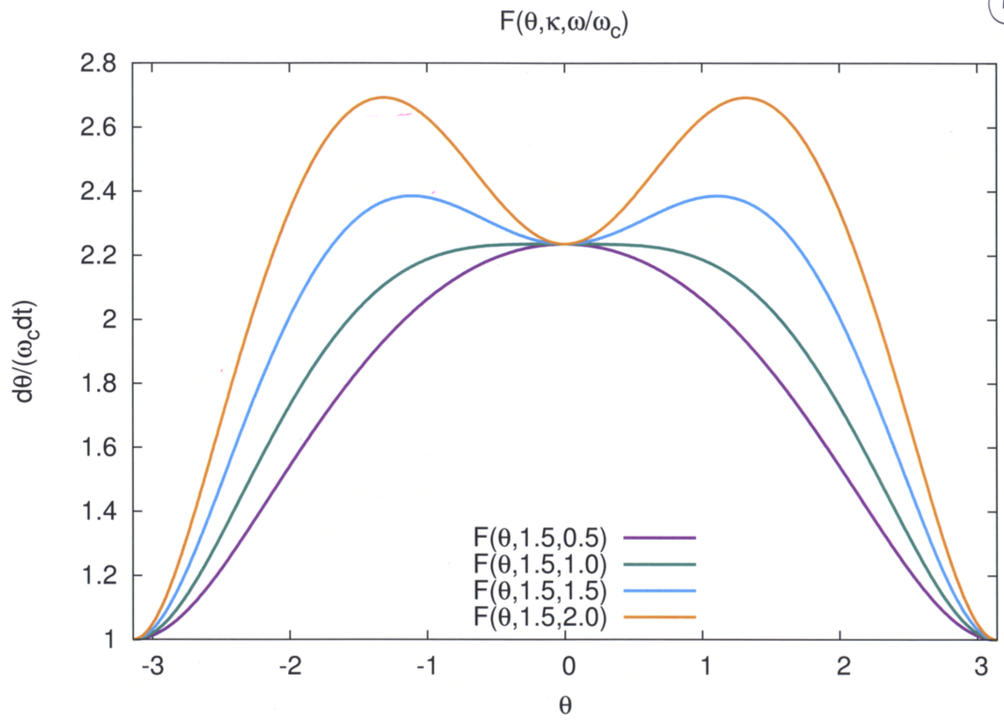
$$\left(\frac{\dot{\theta}}{\omega_c}\right) = \sqrt{2\left\{\kappa + \cos\theta\right\} + \left\{\frac{\omega}{\omega_c}\sin\theta\right\}^2} = F\left(\theta, \kappa, \frac{\omega}{\omega_c}\right)$$

Sjá gröf...

(12a)







(2) Einvíður sveifill með $U = \lambda x^4$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$T = \frac{m}{2}\dot{x}^2, \quad L = \frac{m}{2}\dot{x}^2 - \lambda x^4$$

$$-4\lambda x^3 - m\ddot{x} = 0$$

$$\to \boxed{\ddot{x} + \frac{4\lambda}{m}x^3 = 0}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = p\dot{x} - L = m\dot{x}^2 - \frac{m}{2}\dot{x}^2 + \lambda x^4 = \frac{m}{2}\dot{x}^2 + \lambda x^4$$

$$= \frac{p^2}{2m} + \lambda x^4$$

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\ -\dot{p} &= \frac{\partial H}{\partial x} = 4\lambda x^3 \end{aligned}}$$

hreyfijöfnur Hamiltons fyrir kerfið

(4) Dæmi 7-22 í bók

$$F(x,t) = \frac{k}{x^2}e^{-\frac{t}{\tau}}$$ Kraftur á ögn

finna L og H og skoða varðveislu orku

$$U = \frac{k}{x}e^{-t/\tau} \quad \text{því} \quad F = -\frac{\partial U}{\partial x}$$

$$\boxed{L = \frac{m}{2}\dot{x}^2 - \frac{k}{x}e^{-t/\tau}}, \quad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

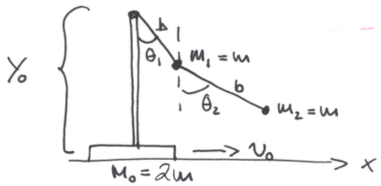
$$H = p\dot{x} - L = m\dot{x}^2 - \frac{m}{2}\dot{x}^2 + \frac{k}{x}e^{-t/\tau}$$

$$= \frac{m}{2}\dot{x}^2 + \frac{k}{x}e^{-t/\tau} = \boxed{\frac{p^2}{2m} + \frac{k}{x}e^{-t/\tau}}$$

$H = T + U \neq$ fasti

því $U \neq U(\dot{x})$

5) Dæmi 7-40 í bók



m_0: x, $y = 0$ } Jafn hraði breytir engu hér

m_1: $x_1 = x + b\sin\theta_1$

$y_1 = y_0 - b\cos\theta_1$

m_2: $x_2 = x_1 + b\sin\theta_2$

$y_2 = y_1 - b\cos\theta_2$

$$\dot x_1 = \dot x + b\cos\theta_1\cdot\dot\theta_1$$
$$\dot y_1 = b\sin\theta_1\cdot\dot\theta_1$$

$$\dot x_2 = \dot x + b\cos\theta_1\cdot\dot\theta_1 + b\cos\theta_2\cdot\dot\theta_2$$
$$\dot y_2 = b\sin\theta_1\cdot\dot\theta_1 + b\sin\theta_2\cdot\dot\theta_2$$

$$U = -mgb\cos\theta_1 - mg\{b\cos\theta_1 + b\cos\theta_2\}$$

þar sem við látum lið með y_0 hverfa

$$T = \frac{2m}{2}\dot x^2 + \frac{m}{2}\left\{\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2\right\}$$

$$= m\dot x^2 + \frac{m}{2}\left\{\dot x^2 + (b\dot\theta_1)^2 + 2b\cos\theta_1\cdot\dot x\dot\theta_1\right\}$$

$$+ \frac{m}{2}\left\{\left[\dot x + b\dot\theta_1\cos\theta_1 + b\dot\theta_2\cos\theta_2\right]^2 + \left[b\dot\theta_1\sin\theta_1 + b\dot\theta_2\sin\theta_2\right]^2\right\}$$

$$L = 2m\dot x^2 + mb^2\dot\theta_1^2 + 2mb\dot x\dot\theta_1\cos\theta_1 + \frac{m}{2}mb^2\dot\theta_2^2 + mb\dot x\dot\theta_2\cos\theta_2$$
$$+ mb^2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2) + 2mgb\cos\theta_1 + mgb\cos\theta_2$$

3 breytur, x, θ_1, og θ_2

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) = 0 \to \frac{d}{dt}\left\{4m\dot x + 2mb\dot\theta_1\cos\theta_1 + mb\dot\theta_2\cos\theta_2\right\} = 0$$

eða

$$4m\ddot x + bm\left\{2\ddot\theta_1\cos\theta_1 + \ddot\theta_2\cos\theta_2\right\} - bm\left\{2\dot\theta_1^2\sin\theta + \dot\theta_2^2\sin\theta_2\right\} = 0$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = 0 \to -2mb\dot x\dot\theta_1\sin\theta_1 - mb^2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2)$$
$$- 2mgb\sin\theta_1 - \frac{d}{dt}\left\{mb^2 2\dot\theta_1 + 2mb\dot x\cos\theta_1 + mb^2\dot\theta_2\cos(\theta_1-\theta_2)\right\} = 0$$

$$0 = \cancel{-2mb\dot x\dot\theta_1\sin\theta_1} \cancel{- mb^2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2)} - 2mgb\sin\theta_1$$
$$- mb^2 2\ddot\theta_1 - 2mb\ddot x\cos\theta_1 \cancel{+ 2mb\dot x\dot\theta_1\sin\theta_1} - mb^2\ddot\theta_2\cos(\theta_1-\theta_2)$$
$$\cancel{+ mb^2\dot\theta_2\dot\theta_1\sin(\theta_1-\theta_2)} - mb^2\dot\theta_2^2\sin(\theta_1-\theta_2)$$

$$\to -2mgb\sin\theta_1 - mb^2 2\ddot\theta_1 - 2mb\ddot x\cos\theta_1 - mb^2\ddot\theta_2\cos(\theta_1-\theta_2)$$
$$- mb^2\dot\theta_2^2\sin(\theta_1-\theta_2) = 0$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = 0$$

$$\to -mg\sin\theta_2 - mb\ddot\theta_2 + m\ddot x\cos\theta_2 + mb\ddot\theta_1\cos(\theta_1-\theta_2) - mb\dot\theta_1^2\sin(\theta_1-\theta_2) = 0$$

6) Dæmi 7-30 í bók

a) $$[g,h] \equiv \sum_k \left\{\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k}\right\}$$ gefið

Finna $$\frac{dg}{dt} = [g,H] + \frac{\partial g}{\partial t}$$

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left\{\frac{\partial g}{\partial q_k}\frac{\partial q_k}{\partial t} + \frac{\partial g}{\partial p_k}\frac{\partial p_k}{\partial t}\right\}$$

nota síðan

$$\frac{\partial q_k}{\partial t} = \dot{q} = \frac{\partial H}{\partial p_k} \quad \text{og} \quad \frac{\partial p_k}{\partial t} = \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Jöfnur Hamiltons

$$\rightarrow \frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}$$