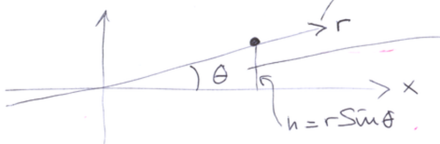
① 7-12 í bók ①

tvívíð slétta, hnit ϕ og r



$$\theta(t) = \alpha t$$

$$v(0) = 0 \quad \text{fyrir ögn}$$

Finna hreyfingu agnar. Ef hnitakerfinu er snúið svona verður engin hreyfing í ϕ-stefnu, enginn kraftur. Það eina sem breytist er $U(r,\theta,t) = mgh = mgr\,\mathrm{Sin}\,\theta(t)$
$= mgr\,\mathrm{Sin}(\alpha t)$

$$T = \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2\right\} = \frac{m}{2}\left\{\dot r^2 + (r\alpha)^2\right\}$$

$$\rightarrow L = \frac{m}{2}\left\{\dot r^2 + (r\alpha)^2\right\} - mgr\,\mathrm{Sin}(\alpha t)$$

eitt alhnit r

②

notum Lagrange

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 0 \rightarrow mr\alpha^2 - mg\,\mathrm{Sin}(\alpha t) - m\ddot r = 0$$

Hreyfijafnan er því

$$\ddot r - r\alpha^2 + g\,\mathrm{Sin}(\alpha t) = 0 \quad \text{eða} \quad \boxed{\ddot r - \alpha^2 r = -g\,\mathrm{Sin}(\alpha t)}$$

Almenna lausn óhliðruðu jöfnunnar er

$$r_h(t) = Ae^{\alpha t} + Be^{-\alpha t}$$

Giskum á sérlausn $r_p(t) = C\,\mathrm{Sin}(\alpha t)$
reynum í hreyfijöfnu

$$-C\alpha^2\,\mathrm{Sin}(\alpha t) - C\alpha^2\,\mathrm{Sin}(\alpha t) = -g\,\mathrm{Sin}(\alpha t)$$

$$\rightarrow 2C\alpha^2 = g \quad \text{eða} \quad C = \frac{g}{2\alpha^2}$$

③

Heildarlausnin er því

$$\boxed{r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2}\,\mathrm{Sin}(\alpha t)}$$

með jaðarskilyrðum

$$\dot r(0) = 0 \qquad r(0) = r_0$$

$$0 = A\alpha - B\alpha + \frac{g}{2\alpha}$$

$$r_0 = A + B$$

eða

$$A - B = -\frac{g}{2\alpha^2}$$
$$A + B = r_0$$

$$\rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{g}{2\alpha^2} \\ r_0 \end{pmatrix}$$

$$\rightarrow \quad A = -\frac{1}{2}\left\{\frac{g}{2\alpha^2} - r_0\right\} = \frac{1}{2}\left\{r_0 - \frac{g}{2\alpha^2}\right\}$$

$$B = \frac{1}{2}\left\{\frac{g}{2\alpha^2} + r_0\right\} = \frac{1}{2}\left\{r_0 + \frac{g}{2\alpha^2}\right\}$$

④

$$\rightarrow r(t) = \frac{1}{2}\left[\left\{r_0 - \frac{g}{2\alpha^2}\right\}e^{\alpha t} + \left\{r_0 + \frac{g}{2\alpha^2}\right\}e^{-\alpha t} + \frac{g}{\alpha^2}\,\mathrm{Sin}(\alpha t)\right]$$

$$= r_0\,\mathrm{Cosh}(\alpha t) + \frac{g}{2\alpha^2}\left\{\mathrm{Sin}(\alpha t) - \mathrm{Sinh}(\alpha t)\right\}$$

Til umhugsunar, ef t.d. $r_0 = 0$ í upphafi hvað gerist þegar $\alpha t > \frac{\pi}{2}$ og þegar $\alpha t \rightarrow \infty$

② dæmi 7-15 í bók

b í jafnvægi

Alhnit gætu verið θ og tímaháð lengd penduls l

$$T = \frac{m}{2}\left\{\dot l^2 + (l\dot\theta)^2\right\}$$

$$U = mgz + \frac{1}{2}k(l-b)^2 = -mgl\,\mathrm{Cos}\theta + \frac{1}{2}k(l-b)^2$$

Lagrange jöfnurnar eru því

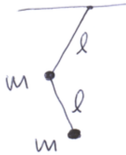
$$\frac{\partial L}{\partial l} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot l}\right) = 0 \;\rightarrow\; ml\dot\theta^2 + mg\cos\theta - k(l-b) - m\ddot l = 0$$

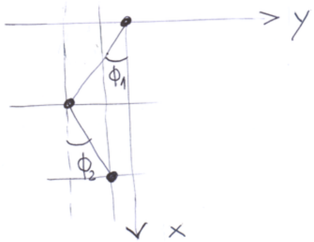
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0 \;\rightarrow\; -mgl\sin\theta - \frac{d}{dt}\left\{2l^2\dot\theta\right\}\frac{m}{2} = 0$$

$$\rightarrow \boxed{\begin{aligned} &\ddot l - l\dot\theta^2 - g\cos\theta + \frac{k}{m}(l-b) = 0 \\ &\ddot\theta + \frac{2\dot l}{l}\dot\theta + \frac{g}{l}\sin\theta = 0 \end{aligned}}$$

③ dæmi 7-07 Tvöfaldur pendúll

sveiflast í sömu sléttu





$$x_1 = l\cos\phi_1 \qquad x_2 = x_1 + l\cos\phi_2$$

$$y_1 = l\sin\phi_1 \qquad y_2 = x_1 + l\sin\phi_2$$

$$T = \frac{m}{2}\left\{\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2\right\}$$

$$= \frac{ml^2}{2}\left\{\dot\phi_1^2 + \dot\phi_1^2 + 2\dot\phi_1\dot\phi_2\left(\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\right) + \dot\phi_2^2\right\}$$

$$= \frac{ml^2}{2}\left\{2\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)\right\}$$

$$U = -mgx_1 - mgx_2$$ ← miðað við okkar hnitakerfi með $U=0$ fyrir $x=0$

$$= -mg\left\{x_1 + x_1 + l\cos\phi_2\right\} = -mgl\left\{2\cos\phi_1 + \cos\phi_2\right\}$$

$$L = T - U = \frac{ml^2}{2}\left\{2\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)\right\} + mgl\left\{2\cos\phi_1 + \cos\phi_2\right\}$$

$$\frac{\partial L}{\partial \phi_1} = -mgl2\sin\phi_1 - \frac{ml^2}{2}2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \dot\phi_1} = 2ml^2\dot\phi_1 + ml^2\dot\phi_2\cos(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \phi_2} = -mgl\sin\phi_2 + ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \dot\phi_2} = ml^2\dot\phi_2 + ml^2\dot\phi_1\cos(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \phi_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_1}\right) = 0 \;\rightarrow\; -mgl2\sin\phi_1 - \cancel{ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)}$$
$$-2ml^2\ddot\phi_1 - ml^2\ddot\phi_2\cos(\phi_1-\phi_2)$$
$$-ml^2\dot\phi_2^2\sin(\phi_1-\phi_2) + \cancel{ml^2\dot\phi_2\dot\phi_1\sin(\phi_1-\phi_2)} = 0$$

$$-gl2\sin\phi_1 - 2l^2\ddot\phi_1 - l^2\ddot\phi_2\cos(\phi_1-\phi_2) - l^2\dot\phi_2^2\sin(\phi_1-\phi_2) = 0$$

$$\rightarrow \boxed{\ddot\phi_1 + \ddot\phi_2\frac{\cos(\phi_1-\phi_2)}{2} + \dot\phi_2^2\frac{\sin(\phi_1-\phi_2)}{2} + \frac{g}{l}\sin\phi_1 = 0}$$

$$\frac{\partial L}{\partial \phi_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_2}\right) = 0 \;\rightarrow\; -mgl\sin\phi_2 + \cancel{ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)}$$
$$-ml^2\ddot\phi_2 - ml^2\ddot\phi_1\cos(\phi_1-\phi_2)$$
$$+ml^2\dot\phi_1^2\sin(\phi_1-\phi_2) - \cancel{ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2)} = 0$$

$$-gl\sin\phi_2 - l^2\ddot\phi_2 - l^2\ddot\phi_1\cos(\phi_1-\phi_2) + l^2\dot\phi_1^2\sin(\phi_1-\phi_2) = 0$$

$$\rightarrow \boxed{\ddot\phi_2 + \underbrace{\ddot\phi_1\cos(\phi_1-\phi_2)}_{\text{hröðunar víxlverkun}} - \underbrace{\dot\phi_1^2\sin(\phi_1-\phi_2)}_{\text{hraða víxlverkun}} + \frac{g}{l}\sin\phi_2 = 0}$$

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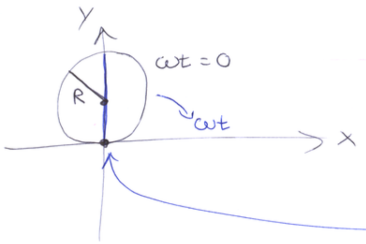
(4) Dæmi 7-11 í bók



Veljum stikun

$$x = R\left\{\cos(\omega t) + \cos(\phi + \omega t)\right\}$$

$$y = R\left\{\sin(\omega t) + \sin(\phi + \omega t)\right\}$$

Ögn á hring sem snýst um punkt á hring, sjá á næstu síðu að þetta er rétt stikun, eða möguleg stikun.

$$\dot{x} = R\left\{-\omega \sin(\omega t) - (\dot{\phi} + \omega)\sin(\phi + \omega t)\right\}$$

$$\dot{y} = R\left\{\omega \cos(\omega t) + (\dot{\phi} + \omega)\cos(\phi + \omega t)\right\}$$

$$L = T = \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2\right\} = \frac{mR^2}{2}\Big[\omega^2 + (\dot{\phi}+\omega)^2 + 2\omega(\dot{\phi}+\omega) \cdot \left[+\sin(\omega t)\sin(\phi+\omega t) + \cos(\omega t)\cos(\phi + \omega t)\right]\Big]$$

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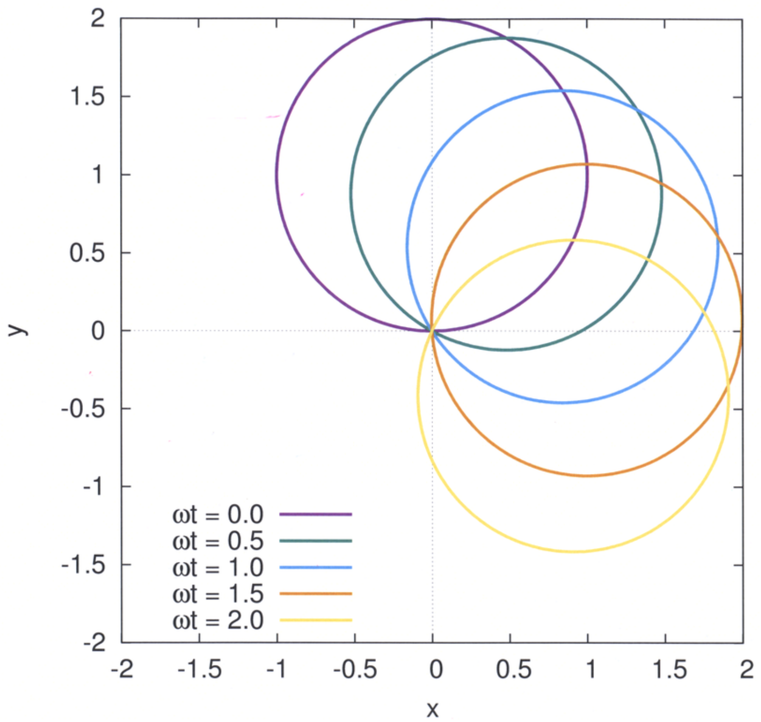

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$$L = \frac{mR^2}{2}\left\{\omega^2 + (\dot{\phi}+\omega)^2 + 2\omega(\dot{\phi}+\omega)\cos\phi\right\}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \rightarrow -\frac{mR^2}{2} 2\omega(\dot{\phi}+\omega)\sin\phi - \frac{d}{dt}\left\{\frac{mR^2}{2} 2(\dot{\phi}+\omega) + 2\omega\cos\phi \cdot \frac{mR^2}{2}\right\} = 0$$

$$\rightarrow -2\omega(\dot{\phi}+\omega)\sin\phi - 2\ddot{\phi} + 2\omega\dot{\phi}\sin\phi = 0$$

$$\rightarrow \boxed{\ddot{\phi} + \omega^2 \sin\phi = 0}$$

hreyfijafna pendúls

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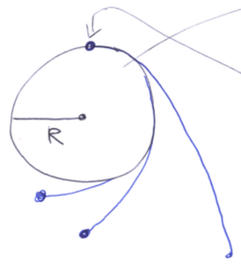
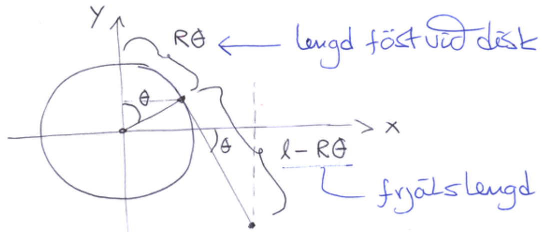
(5) 7-18 í bók

Kyrr diskur, pendúll með lengd l festur við topp hans





$$x = \{l - R\theta\}\cos\theta + R\sin\theta$$

$$y = -\{l - R\theta\}\sin\theta + R\cos\theta$$

$$\dot{x} = -l\dot{\theta}\sin\theta - \cancel{R\dot{\theta}\cos\theta} + R\theta\dot{\theta}\sin\theta + \cancel{R\dot{\theta}\cos\theta} = -l\dot{\theta}\sin\theta + R\theta\dot{\theta}\sin\theta$$

$$\dot{y} = -l\dot{\theta}\cos\theta + \cancel{R\dot{\theta}\sin\theta} + R\theta\dot{\theta}\cos\theta - \cancel{R\dot{\theta}\sin\theta} = -l\dot{\theta}\cos\theta + R\theta\dot{\theta}\cos\theta$$

$$\to T = \frac{m}{2}\left\{\dot x^2+\dot y^2\right\} = \frac{m}{2}\left\{(l\dot\theta)^2+(R\theta\dot\theta)^2-2Rl\theta\dot\theta^2\right\}$$

$$U = mgy = mg\left\{R\cos\theta-(l-R\theta)\sin\theta\right\}$$

$$L = \frac{m}{2}\left\{(l\dot\theta)^2+(R\theta\dot\theta)^2-2Rl\theta\dot\theta^2\right\} - mg\left\{R\cos\theta-(l-R\theta)\sin\theta\right\}$$

Eitt alhnit θ

$$\frac{\partial L}{\partial\theta}-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)=0 \to m(R\dot\theta)^2\theta - mRl\dot\theta^2 + mgR\sin\theta - R\sin\theta\cdot mg + mg(l-R\theta)\cos\theta - \frac{d}{dt}\left\{ml^2\dot\theta+(R\theta)^2m\dot\theta-2Rl\theta\dot\theta m\right\}=0$$

$$m(R\dot\theta)^2\theta - mRl\dot\theta^2 + \cancel{mgR\sin\theta} - \cancel{mgR\sin\theta} + mg(l-R\theta)\cos\theta$$
$$-ml^2\ddot\theta-(R\theta)^2m\ddot\theta-R^2m\dot\theta 2\theta\dot\theta+2mRl\theta\ddot\theta+2Rl\dot\theta^2m=0$$

$$(l-R\theta)^2\ddot\theta-R(l-R\theta)\dot\theta^2-g(l-R\theta)\cos\theta=0$$

$$\to \boxed{(l-R\theta)\ddot\theta-R\dot\theta^2-g\cos\theta=0}$$ er hreyfijafnan

Viljum finna (lítið) horn θ_0 um hvert smáar sveiflur eru samhverfar

$$\theta=\theta_0+\delta$$

$\ddot\theta=\ddot\delta$ $\qquad \cos\theta=\cos(\theta_0+\delta)=\cos\theta_0\cdot\cos\delta-\sin\theta_0\cdot\sin\delta$

$\dot\theta=\dot\delta$

Hreyfijafnan verður þá

$$\left\{l-R(\theta_0+\delta)\right\}\ddot\delta-R\overbrace{\dot\delta^2}^{\sim 0}-g\left\{\cos\theta_0\overbrace{\cos\delta}^{\sim 1}-\sin\theta_0\overbrace{\sin\delta}^{\sim\delta}\right\}=0$$

$$\ddot\delta+\left\{\frac{g\sin\theta_0}{l-R\theta_0}\right\}\delta=\left\{\frac{g\cos\theta_0}{l-R\theta_0}\right\}$$

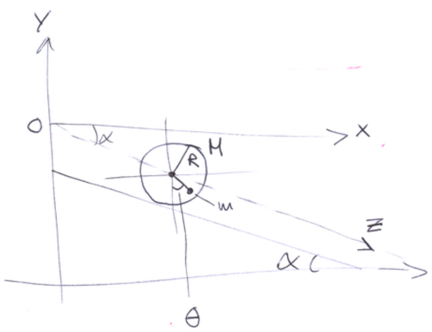
Almenna lausnin fyrir óhliðraða jöfnuna er

$\delta(t)=A\sin(\omega t+\phi_0)$ þar sem

$\omega=\sqrt{\dfrac{g\sin\theta_0}{l-R\theta_0}}$ $\qquad$ Sérlausn má lesa úr jöfnunni $\dfrac{\cos\theta_0}{\sin\theta_0}$

$\to$ Sveiflan er samhverf um θ_0 þegar $\theta_0=\frac{\pi}{2}$ því þá er lausnin aðeins,

6 Dæmi 7-09 í bók



pendúll festur á ás disksins með lengd $l<R$ og massa m

Notum alhnitin z og θ

CM-disks:

$x=z\cos\alpha$

$y=-z\sin\alpha$

$$T=\frac{M}{2}\dot z^2+\frac{I}{2}\dot\phi^2$$

$z=R\phi$

↑ veltiskilyrði

$U=+Mgy$

massi pendúls

$x_b=z\cos\alpha+l\sin\theta$

$y_b=-z\sin\alpha-l\cos\theta$

$$T=\frac{m}{2}\left\{\dot x_b^2+\dot y_b^2\right\}$$

$U=-mgz\sin\alpha-mgl\cos\theta$

I heild

$$T = \frac{M+m}{2}\dot{z}^2 + \frac{I}{2}\dot{\phi}^2 + m\ell^2\dot{\theta}^2 + m\ell\dot{\theta}\dot{z}\,\mathrm{Cos}(\theta+\alpha)$$

$$U = -(M+m)gz\,\mathrm{Sin}\alpha - mg\ell\,\mathrm{Cos}\theta$$

og $I = \frac{MR^2}{2}$, $z = R\phi$

$$\rightarrow L = T - U = \left\{\frac{3M}{4} + \frac{m}{2}\right\}\dot{z}^2 + \frac{m}{2}(\ell\dot{\theta})^2 + m\ell\dot{\theta}\dot{z}\,\mathrm{Cos}(\theta+\alpha)$$
$$- (M+m)gz\,\mathrm{Sin}\alpha + mg\ell\,\mathrm{Cos}\theta$$

alhuitinane z og θ

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \rightarrow \boxed{\ddot{\theta} + \frac{\ddot{z}\,\mathrm{Cos}(\theta+\alpha)}{\ell} + \frac{g}{\ell}\mathrm{Sin}\theta = 0}$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \rightarrow$$

$$\left\{\frac{3M}{2} + m\right\}\ddot{z} - (M+m)g\,\mathrm{Sin}\alpha + m\ell\left\{\ddot{\theta}\,\mathrm{Cos}(\theta+\alpha) - \dot{\theta}^2\,\mathrm{Sin}(\theta+\alpha)\right\} = 0$$