1

Dæmi 1 Ögn í tvívíðri sléttu í mætti $U(x,y) = U_0\left(\frac{xy}{a^2}\right)^2$

a) vel pólhnit sem alhnit, $x = r\cos\theta$, $y = r\sin\theta$

b) $$T = \frac{m}{2}\left[\dot x^2 + \dot y^2\right] = \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2\right\}$$

$$\to \boxed{L = \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2\right\} - U_0\left(\frac{r}{a}\right)^4\cos^2\theta\sin^2\theta}$$

c) $$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = 0$$

$\boxed{4\sin^2\theta\cos^2\theta = \sin^2(2\theta)}$

$$r: \quad -\frac{U_0}{a}\left(\frac{r}{a}\right)^3\sin^2(2\theta) + mr\dot\theta^2 - m\ddot r = 0$$

2

$$\theta: \quad -\frac{U_0}{4}\left(\frac{r}{a}\right)^4 4\sin(2\theta)\cos(2\theta) - \frac{d}{dt}\left\{mr^2\dot\theta\right\} = 0$$

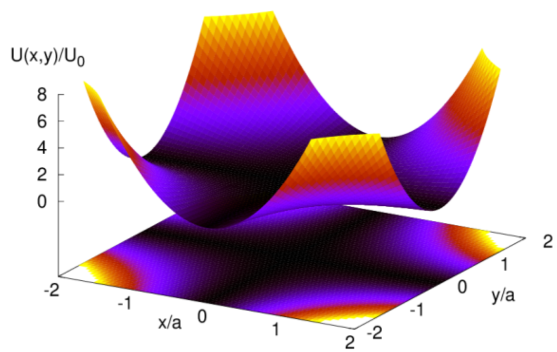
$$\to -U_0\left(\frac{r}{a}\right)^4\sin(2\theta)\cos(2\theta) - mr^2\ddot\theta - m2r\dot r\dot\theta = 0$$

Tökum því saman hreyfijöfnurnar

$$\boxed{\begin{aligned} &\ddot r - r\dot\theta^2 + \frac{U_0}{a}\left(\frac{r}{a}\right)^3\sin^2(2\theta) = 0 \\ &\ddot\theta + \frac{2\dot r\dot\theta}{r} + \frac{U_0}{mr^2}\left(\frac{r}{a}\right)^4\sin(2\theta)\cos(2\theta) = 0 \end{aligned}}$$

d) Mættið U er háð horninu θ, mættið er því ekki miðlægt, og θ er ekki rásuð breyta í falli Lagrange, L. Skoðum mynd af mættinu á næstu síðu

3



Hér sést vel hvernig mættið er hornháð. Eins má lesa úr hreyfijöfnunum að

$$mr^2\dot\theta \neq \text{fasti}$$

e) $$p_r = \frac{\partial L}{\partial \dot r} = m\dot r, \qquad H = p_r\dot r + p_\theta\dot\theta - L$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta, \qquad = \frac{m\dot r^2}{2} + \frac{m}{2}r^2\dot\theta^2 + U_0\left(\frac{r}{a}\right)^4\frac{\sin^2(2\theta)}{4}$$

4

En, $H = H(p,q)$, svo

$$\boxed{H(p_r, p_\theta, r, \theta) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{U_0}{4}\left(\frac{r}{a}\right)^4\sin^2(2\theta)}$$

f) $$\boxed{\begin{aligned} \dot r &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot\theta &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ -\dot p_r &= \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{U_0}{a}\left(\frac{r}{a}\right)^3\sin^2(2\theta) \\ -\dot p_\theta &= \frac{\partial H}{\partial \theta} = U_0\left(\frac{r}{a}\right)^4\sin(2\theta)\cos(2\theta) = \frac{U_0}{2}\left(\frac{r}{a}\right)^4\sin(4\theta) \end{aligned}}$$

5

g) Fall hamiltons og alhnitin eru óháð tíma, orkan er varðveitt, þó hverfipunginn sé ekki varðveittur.

h) Lágmark mættisins $U(x,y)$ myndar kross sem fellur saman við x- og y-ása hnitakerfisins. Því er ljóst að ögn getur ferðast óendanlega langt í burtu frá (0,0) í 4 mismunandi stefnur eftir þessum ásum sé þess gætt að upphafsferð eindarinnar sé ekki 0 og stefnan sé aðeins eftir þeim. Sígild ögn gæti því ferðast eftir einum ásnum frá -óendalegu til óendanlegs án hraðabreytinga og án þess að "verða vör" við hina stefnuna þvert á í miðjunni. Rafeind gæti þetta ekki, hún myndi alltaf "verða vör" við þver stefnuna í miðjunni, og hefði endanlegar líkur á að koma til baka frá krossinum. Inni í krossinum gæti myndast hermuástand rafeindar með mjög langa meðalævi, þetta fer allt eftir orku eindarinnar.

Þessar upplýsingar um sígildu ögnina má lesa beint úr hreyfijöfnunum.

6

Dæmi 2 Braut með $r = a\theta^2$

Notum
$$\underbrace{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{a\theta^2}\right) = \frac{6}{a\theta^4} = \frac{6a}{r^2} \rightarrow \frac{6a}{r^2} + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\rightarrow \frac{6a + r}{r^2} = -\frac{\mu r^2}{\ell^2}F(r) \rightarrow F(r) = -\frac{\ell^2}{\mu}\,\frac{6a+r}{r^4}$$

$$F(r) = -\frac{\partial U}{\partial r} \rightarrow \int_{U(\infty)}^{U(r)} dU = -\int_{\infty}^{r} dr'\, F(r')$$

7

$$\rightarrow U(r) - U(\infty) = +\frac{\ell^2}{\mu}\int_{\infty}^{r} dr' \left[\frac{6a}{r^4} + \frac{1}{r^3}\right] = -\frac{\ell^2}{\mu}\,\frac{r+4a}{2r^3}$$

Setjum $U(\infty) = 0$

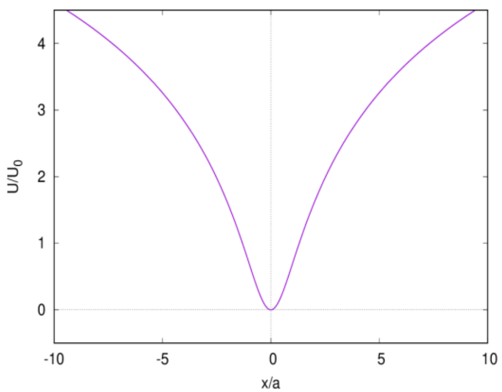
$$\rightarrow \boxed{U(r)} = -\frac{\ell^2}{2\mu}\left[\frac{1}{r^2} + \frac{4a}{r^3}\right]$$

$$= -\frac{\ell^2}{2\mu r^2}\left[1 + \frac{4a}{r}\right]$$

8

Dæmi 3 Einvíð hreyfing í $U(x) = U_0 \ln\left[\left(\frac{x}{a}\right)^2 + 1\right]$

Best er að skoða mættið á mynd



a) Sannfærum okkur að aðeins einn jafnvægispunktur sé til

$$\frac{\partial U(x)}{\partial x} = \frac{2U_0 x}{x^2 + a^2} = 0$$

$$\rightarrow \boxed{x_0 = 0}$$

b)
$$L = \frac{m}{2}\dot{x}^2 - U_0 \ln\left[\left(\frac{x}{a}\right)^2 + 1\right]$$

Notum Euler-Lagrange

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad \rightarrow \quad -\frac{2U_0 x}{x^2+a^2} - m\ddot{x} = 0$$

$$\rightarrow \quad \boxed{\ddot{x} + \frac{2U_0}{m}\,\frac{x}{x^2+a^2} = 0}$$

Setjum $x \simeq x_0 + \delta = \delta, \qquad \delta \ll a$

$$\rightarrow \quad \frac{x}{x^2+a^2} \quad \rightarrow \quad \frac{\delta}{a^2}$$

$$[\omega] = \frac{1}{T}$$

$$\rightarrow \quad \boxed{\ddot{\delta} + \frac{2U_0}{ma^2}\,\delta \simeq 0} \quad \rightarrow \quad \boxed{\omega = \sqrt{\frac{2U_0}{ma^2}}}$$

c) H ? $\qquad H(p,x) = p\dot{x} - L$

$$\rightarrow \quad \boxed{H = \frac{p^2}{2m} + U_0 \ln\left[\left(\frac{x}{a}\right)^2 + 1\right]}$$

Til gamans $\qquad U_0 \ln\left[\left(\frac{x}{a}\right)^2 + 1\right] \simeq U_0 \frac{x^2}{a^2}$

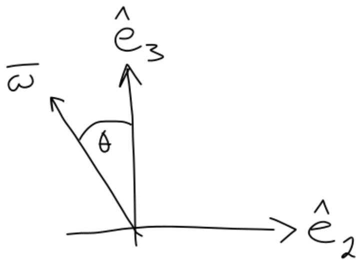
$$\rightarrow \quad H \simeq \frac{p^2}{2m} + \frac{U_0}{a^2}x^2,$$

berist saman við

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$\rightarrow \quad \boxed{\omega = \sqrt{\frac{2U_0}{ma^2}}}$$

Dæmi 4 $\qquad I_{11} = I_{22} = M\left[\frac{5a^2}{8} + \frac{R^2}{2}\right], \quad I_{33} = M\left[\frac{3a^2}{4} + R^2\right]$



$$\rightarrow \bar{\omega} = \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}\omega$$

a) $$\boxed{L = \mathbb{I}\cdot\bar{\omega}} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}\begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}\omega$$

$$= \boxed{\omega\begin{pmatrix} 0 \\ -I_{22}\sin\theta \\ I_{33}\cos\theta \end{pmatrix}}$$

L er í 2-3-sléttu, en almennt ekki samsíða ω

b) $$\boxed{T = \frac{\bar{\omega}^T}{2}\cdot\mathbb{I}\cdot\bar{\omega}}$$

$$= \frac{\omega^2}{2}\left(0, -\sin\theta, \cos\theta\right)\begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}\begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\boxed{= \frac{\omega^2}{2}\left[I_{22}\sin^2\theta + I_{33}\cos^2\theta\right]}$$

Þannig að þegar hornið er 0 fæst orka snúnings um 3-ás, og þegar hornið er 90° fæst snúningur um 2-ásinn eins og vera ber.

(13)

d) Almennur snúningur hnitakerfis λ

$$T = \frac{1}{2}\bar{\omega}^T \cdot \mathbb{I} \cdot \bar{\omega}$$

Snúningur breytir $\boxed{\bar{\omega} \to \lambda\bar{\omega}}$

$$\to (\lambda\bar{\omega})^T = \bar{\omega}^T \lambda^{-1}$$

og $\boxed{\mathbb{I}' = \lambda \mathbb{I} \lambda^{-1}}$

$$T' = \frac{1}{2}(\lambda\bar{\omega})^T \cdot \lambda \mathbb{I} \lambda^{-1} \cdot (\lambda\bar{\omega}) = \frac{1}{2}\bar{\omega}^T \lambda^{-1}\lambda \mathbb{I} \lambda^{-1}\lambda\bar{\omega}$$

$$= \frac{1}{2}\bar{\omega}^T \cdot \mathbb{I}\,\bar{\omega} = T$$

(14)

Dæmi 5

$$T = \frac{m}{2}\left[\dot{x}_1^2 + \dot{x}_2^2\right], \quad U = \frac{k}{2}\left\{13x_1^2 - 12x_1x_2 + 8x_2^2\right\}$$

a)

$$A_{jk} = \frac{\partial^2 U}{\partial x_j \partial x_k}, \quad M_{jk} = \frac{\partial^2 T}{\partial \dot{x}_j \partial \dot{x}_k}$$

$$\to \mathbb{A} = k\begin{pmatrix} 13 & -6 \\ -6 & 8 \end{pmatrix}, \quad \mathbb{M} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\to \mathbb{A}\bar{a} = \omega^2 \mathbb{M}\bar{a} = \omega^2 m\bar{a}$$

sem er hefðbundið eigingildisverkefni

(15)

Eigingildin eru

$$\omega^2 m = \begin{cases} 4k \\ 17k \end{cases} \quad \text{með vigra} \quad \begin{matrix} (1, \frac{3}{2}) \\ (1, -\frac{2}{3}) \end{matrix}$$

sem eru greinilega hornréttir

$$\to \boxed{\omega_1 = \sqrt{\frac{4k}{m}}, \quad \omega_2 = \sqrt{\frac{17k}{m}}}$$

Stöðluðu eignvigrarnir eru því

$$\boxed{\bar{a}_1 = \frac{2}{\sqrt{13}}\left(1, \frac{3}{2}\right), \quad \bar{a}_2 = \frac{3}{\sqrt{13}}\left(1, -\frac{2}{3}\right)}$$

(16)

b)

$$U = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \to U^T \mathbb{A} U = k\begin{pmatrix} 17 & 0 \\ 0 & 4 \end{pmatrix}$$

Normalsveifluhættirnir eru

$$\boxed{\bar{\eta} = U^T\bar{a}} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 & -2 \\ +2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{\frac{1}{\sqrt{13}}\begin{bmatrix} 3x_1 - 2x_2 \\ 2x_1 + 3x_2 \end{bmatrix}}$$

c)

Normalsveifluhættirnir eru hornréttir eiginsveifluhættir með gefna tíðni, því er hægt að örva annan þeirra án þess að hinn fari í gang seinna í tíma.
Sá seinni hér með lægri tíðnina er "samhverfur", en hinn er "andsamhverfur".

d) $\bar{a} = U\bar{\eta}$ (17)

$$X_1(t) = \mathrm{Re}\left\{\frac{3\beta_1}{\sqrt{13}} e^{i\omega_1 t} + \frac{2\beta_2}{\sqrt{13}} e^{i\omega_2 t}\right\}$$

$$X_2(t) = \mathrm{Re}\left[-\frac{2\beta_1}{\sqrt{13}} e^{i\omega_1 t} + \frac{3\beta_2}{\sqrt{13}} e^{i\omega_2 t}\right]$$

þurfum að uppfylla

$X_1(0) = 0$ $\qquad \dot{X}_1(0) = v_0$ $\qquad \beta_i = \beta_i' + i\beta_i''$

$X_2(0) = 0$ $\qquad \dot{X}_2(0) = 0$

$$\left.\begin{aligned} 3\beta_1' + 2\beta_2' &= 0 \\ -2\beta_1' + 3\beta_2' &= 0 \end{aligned}\right\} \rightarrow \boxed{\beta_1',\ \beta_2' = 0}$$

(18)

$$\dot{X}_1(0) = \mathrm{Re}\left\{3i\beta_1'' i\omega_1 + 2i\beta_2'' i\omega_2\right\}\frac{1}{\sqrt{13}} = v_0$$

$$\dot{X}_2(0) = \mathrm{Re}\left[-2i\beta_1'' i\omega_1 + 3i\beta_2'' i\omega_2\right]\frac{1}{\sqrt{13}} = 0$$

$$-3\beta_1''\omega_1 - 2\beta_2''\omega_2 = v_0\sqrt{13}$$

$$2\beta_1''\omega_1 - 3\beta_2''\omega_2 = 0$$

$$\rightarrow \boxed{\beta_1'' = -\frac{3v_0}{\omega_1\sqrt{13}} \qquad \beta_2'' = -\frac{2v_0}{\omega_2\sqrt{13}}}$$