(1)

Dæmi 1 Ögn með föstum hraða v í láréttu x-y sléttunni eftir teini $y = f(x)$, finna skorðukraftana sem verka á hana

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \quad U = 0, \quad L = T$$

Skorður $g(x,y) = y - f(x)$ ← heilnefndar, f(x) hefur vídd L

Kröfuna um fastan hraða er ekki hægt að skrifa sem heilnefnda skorðu

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} \longrightarrow v^2 = \dot{x}^2 + \dot{y}^2$$

$$\longrightarrow 2v\dot{v} = 2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} \longrightarrow \boxed{\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0}$$

Hreyfijöfnurna finnast með og alkraftarnir

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \lambda \frac{\partial g}{\partial q} = 0, \qquad Q_q = \lambda \frac{\partial g}{\partial q}$$

$$q = x, y$$

(2)

því fæst

$$\left.\begin{aligned} -m\ddot{x} - \lambda \frac{\partial f}{\partial x} &= 0 \\ -m\ddot{y} + \lambda &= 0 \end{aligned}\right\} \longrightarrow \begin{aligned} \ddot{x} + \frac{\lambda}{m} f'(x) &= 0 \quad (1) \\ \ddot{y} - \frac{\lambda}{m} &= 0 \quad (2) \end{aligned}$$

og

$$Q_x = \lambda \frac{\partial g}{\partial x} = -\lambda f'(x) = -m\ddot{y} f'(x)$$

$$Q_y = \lambda \frac{\partial g}{\partial y} = \lambda = m\ddot{y}$$

Síðan má nota hraðaskorðurnar til að sjá að

$$Q_y = -m\left(\frac{\dot{x}}{\dot{y}}\right)\ddot{x}$$

$$Q_x = m\left(\frac{\dot{x}}{\dot{y}}\right)\ddot{x} f' = m\ddot{x}\left(\frac{\dot{f}}{\dot{y}}\right)$$

Lengra verður ekki haldið án upplýsinga um f, athuga má vel þekkt tilvik

(3)

Dæmi 2

$$T = \frac{m}{2}\dot{x}^2, \quad U = -\frac{U_0}{\cosh^2(\alpha x)}, \quad L = T - U$$

Viljum finna hreyfijöfnur Hamiltons. Þurfum H með ummyndun Legrende

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = P_x \dot{x} - L = \frac{m}{2}\dot{x}^2 - \frac{U_0}{\cosh^2(\alpha x)}$$

$$= \frac{P_x^2}{2m} - \frac{U_0}{\cosh^2(\alpha x)} = H(P_x, x)$$

$$\dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{m}$$

$$-\dot{P}_x = \frac{\partial H}{\partial x} = 2U_0\alpha \frac{\tanh(\alpha x)}{\cosh^2(\alpha x)}$$

(4)

Dæmi 3 Fall Lagrange: $L = L(q, \dot{q}, \ddot{q}, t)$

Viljum finna jöfnur sambærilegar Euler-Lagrange fyrir þessa tegund L

Skoðum hnikun virkninnar

$$J = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, \ddot{q}, t)$$

Fallið sem við leitum að

$$q(t) = q(0, t)$$

Hnikum um öll möguleg föll η þannig að

$$q(\alpha, t) = q(0, t) + \alpha\eta(t),$$

$$\dot{q}(\alpha, t) = \dot{q}(0, t) + \alpha\dot{\eta}(t)$$

$$\left.\begin{aligned} \eta(t_1) &= 0 \\ \eta(t_2) &= 0 \end{aligned}\right\} \text{Skorður notaðar áður}$$

$$\left.\begin{aligned} \dot{\eta}(t_1) &= 0 \\ \dot{\eta}(t_2) &= 0 \end{aligned}\right\} \text{Viðbótar-skorður}$$

α er þægilegur stiki til að framkvæma hnikunina, en er látinn hverfa að lokum

⑤

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{t_1}^{t_2} dt\, L(q,\dot q,\ddot q,t)$$

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial \alpha} + \frac{\partial L}{\partial \ddot q}\frac{\partial \ddot q}{\partial \alpha}\right]$$

Notum síðan
$$\frac{\partial q}{\partial \alpha} = \eta(t), \quad \frac{\partial \dot q}{\partial \alpha} = \dot\eta(t), \quad \frac{\partial \ddot q}{\partial \alpha} = \ddot\eta(t)$$

$$\rightarrow \frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q}\eta(t) + \frac{\partial L}{\partial \dot q}\dot\eta(t) + \frac{\partial L}{\partial \ddot q}\ddot\eta(t)\right]$$

Tvo fyrstu liðina meðhöndlum við á hefðbundinn hátt eins og í bókinni, sjá líka fyrirlestur 08F síður 3-4

⑥

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q}\eta(t) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\eta(t) + \frac{\partial L}{\partial \ddot q}\ddot\eta(t)\right]$$

Fyrir síðasta liðinn notum við tvöfalda hlutheildun $\left(\int u\,dv = uv - \int v\,du\right)$

$$\frac{\partial L}{\partial \ddot q}\dot\eta(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot q}\right)\dot\eta(t)$$

$$= -\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot q}\right)\eta(t)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\, \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot q}\right)\eta(t)$$

$$\rightarrow \frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot q}\right)\right]\eta(t) = 0$$

⑦

þar sem $\eta(t)$ er almennt fall verður stæðan innan svigans að hverfa

$$\rightarrow \boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot q}\right) = 0} \quad (*)$$

og þessi jafna er útvíkkun á jöfnu Euler og Lagrange

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b)
$$L = -\frac{m}{2}q\ddot q - \frac{k}{2}q^2$$

Beytum (*)

$$-kq - \frac{m}{2}\ddot q + \frac{d^2}{dt^2}\left(-\frac{m}{2}q\right) = 0$$

$$\rightarrow 2m\ddot q + 2kq = 0 \rightarrow \boxed{\ddot q + \frac{k}{m}q = 0}$$

Hreyfijafna fyrir hreintóna sveifil með $\omega_0 = \sqrt{\frac{k}{m}}$

⑧

Dæmi 4

Sveifill með $l = l_0 + a\sin(\omega t)$

$$= l_0\left[1 + \frac{a}{l_0}\sin(\omega t)\right], \quad a < l_0$$

$$T = \frac{m}{2}(\bar v)^2$$

$$\bar v = \dot{\bar r} = \dot l\,\hat e_r + l\dot\theta\,\hat e_\theta, \quad \hat e_r \cdot \hat e_\theta = 0$$

$$U = -mgl\cos\theta, \quad \dot l = \omega a\cos\theta$$

Býst við einu alhniti, θ, og skorðum sem eru ekki heilskorður

Notum Euler-Lagrange

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0 \qquad l = l(t)$$

$$\boxed{L = \frac{m}{2}\left[(l\dot\theta)^2 + \dot l^2\right] + mgl\cos\theta = L(t)}$$

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Því fæst hreyfijafnan

$$-mgl\sin\theta - \frac{d}{dt}\{ml^2\dot{\theta}\} = 0$$

$$gl\sin\theta + 2l\dot{l}\dot{\theta} + l^2\ddot{\theta} = 0$$

$$\boxed{\ddot{\theta} + 2\frac{\dot{l}}{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0}$$

Beitum ummyndun Legendre til að finna H

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$$

$$H = p_\theta\dot{\theta} - L = ml^2\dot{\theta}^2 - \frac{m}{2}l^2\dot{\theta}^2 - \frac{m}{2}\dot{l}^2 - mgl\cos\theta$$

$$= \frac{p_\theta^2}{2ml^2} - \frac{m}{2}\dot{l}^2 - mgl\cos\theta$$

$$= H(p_\theta, \theta, t)$$

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Þannig að

$$\boxed{H = \frac{p_\theta^2}{2ml^2} - \frac{m\dot{l}^2}{2} - mgl\cos\theta}$$

en heildarorkan er

$$\boxed{E = \frac{p_\theta^2}{2ml^2} + \frac{m\dot{l}^2}{2} - mgl\cos\theta}$$

Þessum föllum ber því ekki saman. Fyrir tímaháð kerfi er fall Hamiltons ekki jafnt heildarorkunni

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Hreyfijöfnur Hamiltons

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial\theta} = mgl\sin\theta$$

$$l = l_0\left(1 + \frac{a}{l_0}\sin(\omega t)\right)$$

Athyglisvert er hve einfaldar þessar hreyfijöfnur eru. Í þeim kemur ekki fyrir tímaafleiðan af lengdinni l. Hún kæmi vissulega fyrir ef við setjum þær saman í eina annarsstigs afleiðujöfnu